

# FREE VIBRATIONS OF LAMINATED COMPOSITE PLATES USING LAYERWISE DISPLACEMENT MODEL

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**Abstract:** *In this paper Generalized Layer Wise Plate Theory of Reddy (GLPT) is used to formulate an isoparametric finite element model for free vibration of laminated composite plates. With the assumed displacement field, linear strain displacement relations and linear elastic orthotropic material properties for each lamina, virtual work statement is utilized in order to formulate isoparametric finite element model. The original MATLAB computer program is coded for finite element solution. Some new results using GLPT finite element model for soft core sandwich plate is presented, which may be used as the guideline for their optimal design in the laboratory.*

**Key words:** *Free vibrations, Sandwich plate, Finite element*

## 1. Introduction

Structural members made of fiber reinforced laminated composite and sandwich plates, like those in automobile, bridge, submarine and aircraft industry, are often subjected to dynamic loads. In structural applications two types of dynamical behavior are of primary importance: free vibrations and forced response. Free vibrations are the motions resulting from specified initial conditions in the absence of applied loads, while forced vibrations are the motion resulting from specified inputs to the system from external sources [1]. Since, forced vibrations response strongly depend on the values of free vibration parameters, like natural frequencies and mode shapes of vibration, and for a given amplitude of

loading, dynamic response may sometimes be greater than static response, in this paper only free vibrations are considered.

The accuracy in obtaining free vibration frequencies of composite plates is closely related to the assumed shear deformation pattern. It has been shown that Equivalent Single Layer (ESL) theories yield good predictions when materials properties of adjacent layers do not differ significantly. However, since they use continuously differentiable function of thickness coordinate, they are unable to account for severe discontinuities in transverse shear strains that occur at the interfaces between the layers with drastically different stiffness properties. In these cases, the local deformations and stresses, and sometimes even the overall laminate response, such as fundamental frequencies are not well predicted.

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In wish to overcome the shortcomings of ESL theories, and reduce the computational cost of 3D elasticity theory, discrete layer or layer wise (LW) theories have been proposed. These theories are based on unique displacement field for each layer and with the use of post processing procedure may enforce the interlaminar continuity of transverse shear stresses.

In this paper a discrete layer theory called Generalize Layerwise Plate Theory (GLPT) of Reddy is used to study free vibrations of laminated composite and sandwich plates. The theory assumes layer wise variation of in-plane displacements and constant transverse displacement. The resulting strain field is kinematically correct in that the in-plane strains are continuous through the thickness, while the transverse shear strains are discontinuous through the thickness, allowing for the possibility of continuous transverse shear stresses [3]. Transverse shear stresses satisfy Hook's low, 3D equilibrium equations, interlaminar continuity and traction free boundary conditions and have quadratic variation within each layer of the laminate. Using assumed displacement field, linear strain displacement relations and 3D constitutive equations of lamina, governing differential equations of motion are derived using Hamilton's principle. Virtual work statement is then utilized to formulate isoparametric finite element model (FEM).

The objective of this paper is to code a MATLAB computer program for FEM solutions based on GLPT, capable of calculating fundamental frequencies of laminated composite and sandwich plates. The accuracy of computer program will be verified by comparison with available results from the literature.

## 2. Theoretical formulation

### 2.1. Displacement field

A laminated plate composed of  $n$  orthotropic lamina is shown on Figure 1. The integer  $k$  denotes the layer number that starts from the plate bottom. Plate middle surface coordinates are  $(x, y, z)$ , while layer coordinates are  $(x_k, y_k, z_k)$ . Plate and layer thickness are denoted as  $h$  and  $h_k$ , respectively. It is assumed that 1) layers are perfectly bonded together, 2) material of each layer is linearly elastic and has three planes of materials symmetry (i.e., orthotropic), 3) strains are small, 4) each layer is of uniform thickness and 5) inextensibility of normal is imposed.

The displacements components  $(u_1, u_2, u_3)$  at a point  $(x, y, z)$  can be written as:

$$\begin{aligned} u_1(x, y, z) &= u(x, y) + \sum_{I=1}^N U^I(x, y) \cdot \Phi^I(z), \\ u_2(x, y, z) &= v(x, y) + \sum_{I=1}^N V^I(x, y) \cdot \Phi^I(z), \\ u_3(x, y, z) &= w(x, y). \end{aligned} \quad (1)$$

where  $(u, v, w)$  are the displacements of a point  $(x, y, 0)$  on the reference plane of the laminate,  $U^I$  and  $V^I$  are undetermined coefficients, and  $\Phi^I(z)$  are layerwise continuous functions of the thickness coordinate. In this paper a linear Lagrange interpolation of in-plane displacement components through the thickness is assumed.

### 2.2. Strain displacement relations

The strains associated with the displacement field (1) can be computed using linear strain-displacement relation:

$$\begin{aligned}\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \sum_{I=1}^N \frac{\partial U^I}{\partial x} \Phi^I, \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} + \sum_{I=1}^N \frac{\partial V^I}{\partial y} \Phi^I, \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \sum_{I=1}^N \left( \frac{\partial U^I}{\partial y} + \frac{\partial V^I}{\partial x} \right) \Phi^I, \\ \gamma_{xz} &= \sum_{I=1}^N U^I \frac{d\Phi^I}{dz} + \frac{\partial w}{\partial x}, \\ \gamma_{yz} &= \sum_{I=1}^N V^I \frac{d\Phi^I}{dz} + \frac{\partial w}{\partial y}.\end{aligned}\quad (2)$$

### 2.3. Constitutive relations

$$\{\boldsymbol{\sigma}\}^{(k)} = [\mathbf{Q}]^{(k)} \cdot \{\boldsymbol{\varepsilon}\}^{(k)} \quad (3)$$

$$\begin{aligned}& \int_0^t \int_{\Omega} \left\{ N_{xx} \frac{\partial \delta u}{\partial x} + N_{yy} \frac{\partial \delta v}{\partial y} + N_{xy} \left( \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) + Q_x \frac{\partial \delta w}{\partial x} + Q_y \frac{\partial \delta w}{\partial y} + \right. \\ & + \sum_{I=1}^N \left[ N_{xx}^I \frac{\partial \delta U^I}{\partial x} + N_{yy}^I \frac{\partial \delta V^I}{\partial y} + N_{xy}^I \left( \frac{\partial \delta U^I}{\partial y} + \frac{\partial \delta V^I}{\partial x} \right) + Q_x^I U^I + Q_y^I V^I \right] + \\ & - I_0 (\ddot{u} \delta u + \ddot{v} \delta v + \ddot{w} \delta w) - \sum_{I=1}^N I^I (\ddot{U}^I \delta u + \ddot{u} \delta U^I + \ddot{V}^I \delta v + \ddot{v} \delta V^I) - \\ & \left. - \sum_{I=1}^N \sum_{J=1}^N I^{IJ} (\ddot{U}^I \delta U^J + \ddot{V}^I \delta V^J) \right\} d\Omega dt = 0.\end{aligned}\quad (4)$$

where

$$I_0 = \int_{-h/2}^{h/2} \rho dz, I^I = \int_{-h/2}^{h/2} \rho \Phi^I dz, I^{IJ} = \int_{-h/2}^{h/2} \rho \Phi^I \Phi^J dz$$

and  $\rho$  is mass density.

### 3. Finite Element Model

The GLPT finite element requires only the  $C^0$  continuity of major unknowns, thus in each node only a displacement components are adopted, that is  $(u, v, w)$  in

where:

$\boldsymbol{\sigma}^{(k)} = \{\sigma_{xx} \ \sigma_{yy} \ \tau_{xy} \ \tau_{xz} \ \tau_{yz}\}^{(k)T}$  and  $\boldsymbol{\varepsilon}^{(k)} = \{\varepsilon_{xx} \ \varepsilon_{yy} \ \gamma_{xy} \ \gamma_{xz} \ \gamma_{yz}\}^{(k)T}$  are stress and strain components respectively, and  $\mathbf{Q}_{ij}^{(k)}$  are transformed elastic coefficients, of k-th lamina in global coordinates [3].

### 2.4. Virtual work statement

The virtual work statement can be written using Hamilton's principle, by neglecting the body forces as:

the middle surface element nodes and  $(U^I, V^I)$  in the I-th plane element nodes, Figure 1.

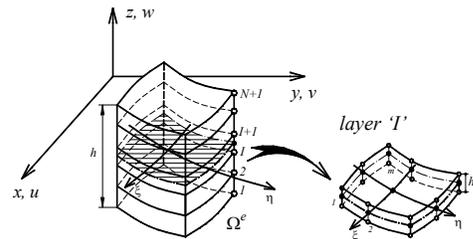


Fig. 1. Plate finite element with  $n$  layers and  $m$  nodes

### 3.1 Displacement field

The generalized displacements over element  $\Omega^e$  can be expressed as:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}^e = \begin{Bmatrix} \sum_{j=1}^m u_j \Psi_j \\ \sum_{j=1}^m v_j \Psi_j \\ \sum_{j=1}^m w_j \Psi_j \end{Bmatrix}^e = \sum_{j=1}^m [\Psi_j]^e \{\mathbf{d}_j\}^e$$

$$\begin{Bmatrix} U^I \\ V^I \end{Bmatrix}^e = \begin{Bmatrix} \sum_{j=1}^m U_j^I \Psi_j \\ \sum_{j=1}^m V_j^I \Psi_j \end{Bmatrix}^e = \sum_{j=1}^m [\bar{\Psi}_j]^e \{\mathbf{d}_j^I\}^e \quad (5)$$

where:

$$\{\mathbf{d}_j\}^e = \{u_j^e \quad v_j^e \quad w_j^e\}^T, \{\mathbf{d}_j^I\}^e = \{U_j^I \quad V_j^I\}^T$$

are displacement vectors in the middle plane and I-th plane, respectively and  $\Psi_j^e$  are interpolation functions, for the j-th node of the element  $\Omega^e$ , while  $[\Psi_j]^e$  and  $[\bar{\Psi}_j]^e$  are given in [9].

### 3.2 Free vibration analysis

Substituting assumed displacement field (5) into equation (4) the finite element model is obtained:

$$[\mathbf{M}]^e \{\ddot{\Delta}\}^e + [\mathbf{K}]^e \{\Delta\}^e = 0 \quad (6)$$

where element stiffness and element mass matrix are given in [5]. Solution of equations (6) gives eigenvalues or vibration frequencies  $\omega_1, \omega_2, \dots, \omega_N$ . The smallest of vibration frequencies not equal to zero is the critical frequency  $\omega_{cr}$  and

the corresponding eigenfunctions are mode shapes.

## 4. Numerical results and discussion

Using previous derived finite element solutions, a MATLAB computer program was coded, for free vibration of laminated composite and sandwich plates. Element stiffness and mass matrix were evaluated using 2x2 and 3x3 Gauss-Legendre integration scheme, for 2D linear and quadratic in-plane interpolation, respectively. Consistent element mass matrix was implemented, in order to preserve the total mass of the element [2]. The accuracy of present formulation is demonstrated by comparison with results available from the literature.

(6)

### 4.1 Example of cross ply and angle ply laminates

For simply supported angle-ply and cross-ply square laminated plates ( $a/h=10$ ), composed of layers with equal thickness and made of following material:

$$\begin{aligned} E_1/E_2 = 25, G_{12}/E_2 = 0.5, G_{13}/E_2 = 0.5, \\ G_{23}/E_2 = 0.2, \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \end{aligned} \quad (7)$$

effects of lamination schemes on the fundamental frequency are analyzed and presented in Table 1. It may be seen that for thin laminates all three theories (FSDT [4], HSDT [4] and GLPT (Present)) give accurate prediction, while for thick laminates some discrepancy is observed for FSDT.

Effects of lamination schemes on the fundamental frequency of simply supported square laminate plate

Table 1

Scheme	a/h=4			a/h=10			a/h=100		
	FSDT <sup>[4]</sup>	HSDT <sup>[4]</sup>	Present	FSDT <sup>[4]</sup>	HSDT <sup>[4]</sup>	Present	FSDT <sup>[4]</sup>	HSDT <sup>[4]</sup>	Present
0/0/0	7.6590	7.7390	8.0047	12.4540	12.4650	12.7288	15.1930	15.1930	15.1926
0/30/0	7.6350	7.5730	7.8448	12.4910	12.3800	12.5200	15.3570	15.3530	15.2074
0/45/0	7.5810	7.4130	7.5952	12.4500	12.2130	12.3413	15.4070	15.4000	15.2391
0/60/0	7.5030	7.2580	7.3350	12.3370	12.0050	12.0367	15.3480	15.3400	15.2226
0/90/0	7.4140	7.1230	7.0867	12.1650	11.7580	11.6724	15.1870	15.1770	15.2431

#### 4.2 Example of sandwich plate

A five layer (0/90/core/0/90) symmetric simply supported sandwich plate is analyzed in the following example, Table 2,3 and Figure 1. Cross-ply face sheets are each of thickness  $t_f$ , while core is of thickness  $t_c$ , and are made of following material:

Face sheets (Graphite-Epoxy T300/934):

$$E_1 = 131 \text{ GPa}, E_2 = E_3 = 10.34 \text{ GPa},$$

$$G_{12} = G_{23} = 6.895 \text{ GPa}, G_{13} = 6.205 \text{ GPa},$$

$$\nu_{12} = \nu_{13} = 0.22, \nu_{23} = 0.49, \rho = 1627 \text{ kg/m}^3$$

Core (Isotropic):

$$E_1 = E_2 = E_3 = 6.89 \times 10^{-3} \text{ GPa},$$

$$G_{12} = G_{23} = G_{13} = 3.45 \times 10^{-3} \text{ GPa},$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0, \rho = 97 \text{ kg/m}^3$$

Free vibrations are normalized in the form:

$$\bar{\omega} = \omega b^2 / h \sqrt{(\rho / E_2)_f} \quad (8)$$

The accuracy of the proposed GLPT model is verified calculating fundamental frequencies of soft-core sandwich plates for various parameters, like: aspect ratio (a/b) in Table 2, ratio of core to face thickness ( $t_c/t_f$ ) in Table 3 and side to thickness ratio (a/h) on Figure 2.

Fundamental frequencies of sandwich plate a/b=1 and  $t_c/t_f=10$

Table 2

a/b	Present	HSDT <sup>[4]</sup>	HSDT <sup>[8]</sup>	HSDT <sup>[7]</sup>
0.5	5.7798	15.0316	21.4500	39.4840
1	1.8671	4.8594	7.0473	13.8694
1.5	1.1089	2.8188	4.1587	9.4910
2	0.8255	2.4560	3.6444	10.1655
2.5	0.6854	1.5719	2.3324	6.5059
3	0.6047	1.3040	1.9242	5.6588
5	0.4758	0.8187	1.1541	3.6841

Fundamental frequencies of sandwich plate a/b=1 and a/h=10

Table 3

$t_c/t_f$	Present	HSDT <sup>[4]</sup>	HSDT <sup>[4]</sup>	HSDT <sup>[4]</sup>
4	1.9846	8.9948	10.7409	13.9190
10	1.8671	4.8594	7.0473	13.8694
20	2.1364	3.1435	4.3734	12.8946
30	2.3356	2.8481	3.4815	11.9760
40	2.4719	2.8266	3.1664	11.2036
50	2.5687	2.8625	3.0561	10.5557
100	2.7921	3.0290	3.0500	8.4349

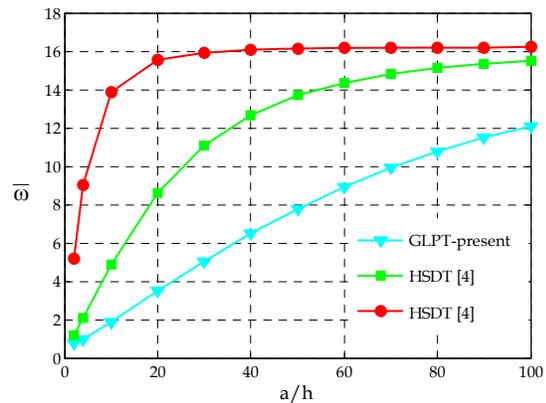


Fig. 2. Fundamental frequencies of thin thick and thin sandwich plates

The obtained results in Tables 2,3 and Figure 2 imply that ESL theories are unable to accurately predict fundamental frequencies of soft-core sandwich plates, since their assumed displacement field can not account for significant change of materials properties between adjacent layers.

#### 4. Conclusions

In this paper the finite element solutions are presented for free vibration analysis of laminated composite and sandwich plates, using GLPT. The study has verified that the proposed model is capable to accurately predict fundamental frequencies of laminated composite and sandwich plates, while ESL theories are inapplicable for the analysis of highly anisotropic laminates. Namely, thin plate assumption increases the stiffness of the plate and therefore yields to lower deflections and higher fundamental frequencies. HSDT is however able to accurately predict global response of general laminated plates, but its deviations from LW solutions increases for sandwich plates. Finally, present layerwise finite element model has shown the importance of different parameters on natural frequencies of both thin and thick laminated composite and sandwich plate, which may be used as the guideline for their optimal design in the laboratory.

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