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ACCURACY ANALYSIS OF EXTENDED FINITE ELEMENT METHOD FOR TENSION OF PLATE WITH A CENTRAL CRACK

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Abstract: *The accuracy of the Extended Finite Element Method (XFEM) in solving problems of straining plate with central crack is analyzed. Numerical solutions for displacements and stresses are compared with analytical solutions, while numerical solutions for stress intensity factor are compared with empirical solutions. The impact of the mesh density and near tip (NT) enrichment function are investigated, as well as the impact of the size of the integration domain of J integral used to calculate stress intensity factor* K_I *. The use of fine mesh and the NT enrichment functions increase the accuracy of the solution.*

Key words: *accuracy analysis, XFEM, mesh density, NT functions, stress intensity factor.*

1. Introduction

The extended finite element method, XFEM, reduce computational fracture mechanics problems by not requiring the finite element mesh to coincide with cracks. XFEM is based on using enrichment functions at finite elements that are crossed by the crack. Basis of the XFEM method developed by Belytchko and Black [1], were presented in [4] for the two-dimensional cracks.

Creating a model using XFEM is divided into two steps [3-6]: generation of the mesh for the geometric domain without including cracks, and enriching the finite element approximation by additional functions that model the cracks, without explicitly separating sides of the cracks. There are two enriching functions in XFEM for fracture mechanics: discontinuity Heaviside step functions (H) and asymptotic near tip functions (NT).

The goal of this paper is to analyze the accuracy of the Extended finite element method for 2D problem, by comparing the obtained numerical results with analytical and empirical solutions.

2. Approximation in XFEM

The approximation of displacement in the XFEM is decomposed into a continuous and enrichment part, as

$$u(x) = u_{con}(x) + u_{enrh}(x) \qquad (1)$$

where the first addend is standard approximation in the FEM, and second is enrichment part of displacement approximation around the crack. The

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region of the crack for enrichment by H and NT functions is shown in Figure 1 [5].

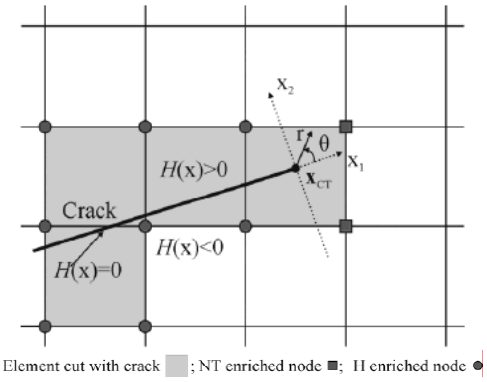
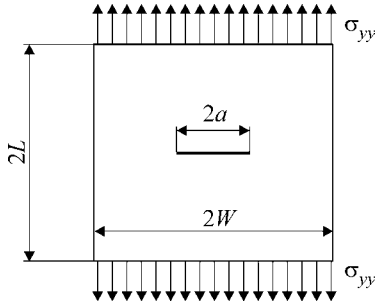


Fig. 1. *Enriched region of a crack*

3. Numerical experiments

Numerical experiments which aimed to demonstrate the impact of mesh parameters and types of nodes enrichment on the accuracy of numerical solutions were performed. Those experiments were performed on a test example of the plate with a central crack, as an example that has strong open discontinuities in fields of displacements. Data for the geometry, material parameters and load are given in Figure 2.



$\sigma_{yy} = 6.89 KPa,\ W = 254\ mm,\ L = 127\ mm$
$a = 127\ mm,\ E = 2.068 \times 10^5\ MPa,\ \nu = 0.3$
$t = 25.4\ mm$

Fig. 2. *Plate with a central crack*

Using the symmetry conditions, only the right half of the plate was modeled using the XFEM method [4]. The following effects were investigated: 1) density of the mesh, 2) enrichments with NT functions around crack tip, 3) the size of the integration domain of J integral, used to calculate stress intensity factor K_I. The plate was modeled with 10x10, 20x20, 40x40 and 80x80 elements. The influence of enrichment functions around crack tip was analyzed in two ways: 1) nodes of elements that are crossed by the crack are enriched with Heaviside function and nodes around the crack tip are enriched with NT (H+NT), 2) nodes of elements that are crossed by the crack are enriched only with Heaviside function (H).

In examining the impact of integration domain of J integral, the following relations between radius and crack length were used r / a (%) : 10, 15, 20, 25, 30, 35, 40 и 45%. One integration domain is given in Figure 3, [4].

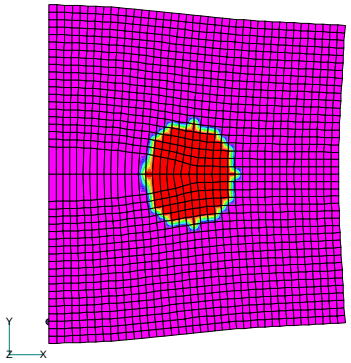


Fig. 3. *One of the integration domains of J integral*

For all these variants of calculations, the values of the stress intensity factor K_I are calculated, and the results are shown in Table 1. Numerical results for the stress intensity factor K_I are compared with the empirical value. The empirical value was obtained from the relation:

$$K_I^{emp} = \sigma F\left(\frac{a}{W}, \frac{L}{W}\right)\sqrt{\pi a} \qquad (2)$$

and, based on the applied load and chosen value of the correction factor (Figure 3.11 in [2]), amounts

$$K_I^{emp} = 261.08 KPa\sqrt{mm}\,. \qquad (3)$$

For each obtained numerical value of stress intensity factor, there is a calculated error Er(%) given in Table 1. It was noticed that in case of coarse mesh the size of the integration domain should be considered, because it can happen that a small integration domain includes only a small number of nodes, and therefore get an error over 10%, so that result should be discarded as incorrect. In all other cases, the good agreement is obtained between numerical and empirical results. Increasing mesh density reduces the error and rapid convergence of the solution is apparent. Enrichment around crack tip with NT functions provides very good results even with coarse mesh. In all the cases when the NT enrichment was used, the error is below 3%. In cases with better mesh density can be noticed that the obtained numerical results are above the theoretical, leading to the conclusion that empirical solution read from the diagram given in the literature is not accurate enough.

Stress intensity factor K_I *(10x10; 20x20; 40x40; 80x80)* Table 1

$\frac{r}{a}$(%)	K_I XFEM(H) 10x10	Er(%)	K_I XFEM(H+NT) 10x10	Er(%)
10	*121.15*	*53.59*	137.88	47.19
15	*121.15*	*53.59*	137.88	47.19
20	*121.15*	*53.59*	170.28	34.78
25	*204.77*	*21.57*	225.20	13.74
30	241.27	7.59	255.73	2.05
35	241.27	7.59	255.73	2.05
40	241.27	7.59	257.78	1.26
45	241.20	7.61	255.51	2.13
M.v.	191.66	26.59	212.00	18.80

$\frac{r}{a}$(%)	K_I XFEM(H) 20x20	Er(%)	K_I XFEM(H+NT) 20x20	Er(%)
10	114.54	56.13	172.59	33.89
15	254.15	2.65	262.27	0.46
20	254.15	2.65	263.80	1.04
25	253.90	2.75	261.74	0.25
30	254.38	2.57	264.72	1.39
35	252.60	3.25	260.36	0.28
40	255.32	2.21	263.15	0.79
45	253.93	2.74	261.45	0.14
M.v.	236.62	9.37	251.26	3.76

$\frac{r}{a}$(%)	K_I XFEM(H) 40x40	Er(%)	K_I XFEM(H+NT) 40x40	Er(%)
10	257.51	1.37	266.41	2.04
15	262.41	0.51	267.64	2.51
20	262.18	0.42	266.46	2.06
25	260.97	0.04	266.57	2.10
30	260.84	0.09	265.53	1.71
35	260.17	0.35	264.45	1.29
40	260.43	0.25	264.35	1.25
45	260.00	0.41	263.90	1.08
M.v.	260.56	0.20	265.66	1.76

$\frac{r}{a}$(%)	K_I XFEM(H) 80x80	Er(%)	K_I XFEM(H+NT) 80x80	Er(%)
10	264.81	1.43	267.93	2.62
15	265.06	1.52	267.49	2.46
20	264.59	1.35	266.70	2.15
25	263.90	1.08	266.32	2.01
30	263.13	0.79	265.33	1.63
35	262.91	0.70	264.98	1.49
40	262.63	0.59	264.60	1.35
45	262.28	0.46	264.23	1.21
M.v.	263.66	0.99	265.95	1.86

After determining empirical values of the stress intensity factor, analytical solutions for stresses and displacements around the crack tip are calculated using the Table 2.

Analytical solutions for stresses and displacements Table2

	Crack mode I
u_1	$\frac{K_1}{2\mu}\sqrt{r/(2\pi)}\cdot\cos\frac{\theta}{2}\left[\kappa - 1 + 2\sin^2\frac{\theta}{2}\right]$
u_2	$\frac{K_1}{2\mu}\sqrt{r/(2\pi)}\cdot\sin\frac{\theta}{2}\left[\kappa + 1 - 2\cos^2\frac{\theta}{2}\right]$
σ_{11}	$K_1/\sqrt{2\pi r}\cdot\cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$
σ_{22}	$K_1/\sqrt{2\pi r}\cdot\cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$
σ_{12}	$K_1/\sqrt{2\pi r}\cdot\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$

In the following analysis, numerically obtained stress values in the center of the element and analytically calculated values

for the same position were compared. Element, whose node was in the crack tip, was analyzed and obtained results are shown in Table 3 for different mesh densities and different enrichment functions around the crack tip.

The best matching results in case of enriching with NT functions are obtained with σ_{yy} component of stress, where the error in all variants of mesh density does not exceed 5%. Elements enriched with NT functions around the crack tip give a more accurate approximation of the stress field and a better agreement with the analytical results.

Stress values in the center of the element (10x10; 20x20; 40x40; 80x80) Table 3

Stress	Analytical result	XFEM(H) 10x10	XFEM(H+NT) 10x10
σ_x (error)	14.68	8.00 (45.5%)	8.82 (39.9%)
σ_y (error)	30.73	24.85 (19.1%)	30.21 (1.7%)
σ_{xy} (error)	3.33	3.36 (1.05%)	1.26 (62.1%)

Stress	Analytical result	XFEM(H) 20x20	XFEM(H+NT) 20x20
σ_x (error)	20.76	15.28 (26.4%)	16.20 (22%)
σ_y (error)	43.46	37.61 (13.5%)	44.33 (2%)
σ_{xy} (error)	4.70	5.19 (10.4%)	2.01 (57.2%)

Stress	Analytical result	XFEM(H) 40x40	XFEM(H+NT) 40x40
σ_x (error)	29.36	25.79 (12.1%)	26.82 (8.6%)
σ_y (error)	61.47	54.96 (10.6%)	63.73 (3.7%)
σ_{xy} (error)	6.65	7.62 (14.6%)	2.98 (55.2%)

Stress	Analytical result	XFEM(H) 80x80	XFEM(H+NT) 80x80
σ_x (error)	41.52	40.62 (2.2%)	41.79 (0.65%)
σ_y (error)	86.93	78.93 (9.2%)	90.75 (4.4%)
σ_{xy} (error)	9.41	10.91 (15.9%)	4.25 (54.8%)

In further analysis stress values according to the distance of the center of the element from the crack tip were compared. Figure 4 shows the comparative diagrams of normal stress σ_{yy} depending on the distance from the crack tip for different mesh densities, as well as a diagram of analytical solution. Based on the obtained results it can be concluded that the calculation results have a good agreement in the area around the crack tip for all sizes of mesh. All calculations show the character of the rapid growth of stress when asymptotically approaching the top of the crack. Because of the large impact of the distance from the crack tip to the center of the element to the value of the maximum stress, it is recommended to use a finer mesh around the crack tip.

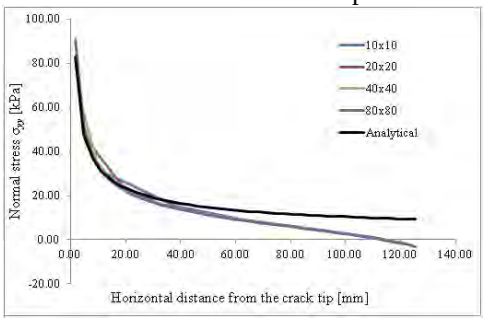


Fig. 4. *Normal stress σ_{yy} depending on the distance of the center of the element from the crack tip for different mesh densities*

Figure 5 shows the comparative diagrams of normal stress σ_{yy} depending on the distance of the center of the element

to the crack tip for the mesh density of 80x80 with enrichments H and H+NT. Based on the obtained results it can be concluded that the results for normal stresses are in agreement with the analytical result in both variants of enrichment. Stress values obtained enriching with NT functions give the expected slightly higher stresses, than the variant that uses only H functions.

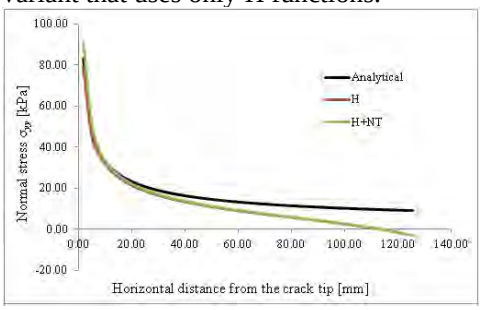


Fig. 5. *Normal stress σ_{yy} depending on the distance of the center of the element to the crack tip for the mesh density of 80x80 with enrichments H and H+NT*

Figure 6 shows comparative diagrams of the displacements of nodes in x direction depending on their vertical distance from a crack tip for mesh density 80x80, and also a diagram of the analytical solution obtained according to Table 2.

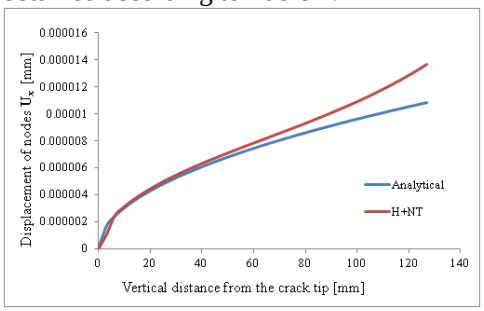


Fig. 6. *Displacements of nodes in x direction depending on their vertical distance from a crack tip for mesh density 80x80*

Figure 7 shows comparative diagrams of the displacements of nodes in y direction depending on their vertical distance from the crack tip for mesh density 80x80, as well as the diagram for the analytical solution according to Table 2.

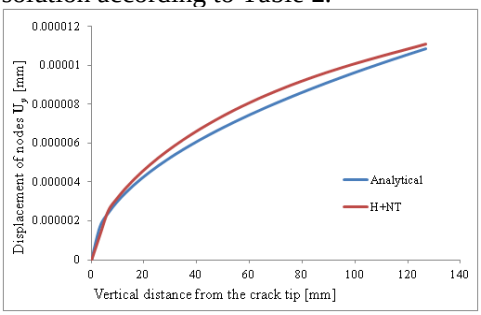


Fig. 7. *Displacements of nodes in y direction depending on their vertical distance from the crack tip for mesh density 80x80*

It can be concluded from the results that there is a good agreement between numerical and analytical results for the displacements of nodes in y direction depending on the vertical distance from the crack tip, as shown in Figures 6 and 7.

Impact of different enrichment around the crack tip for 10x10 mesh is shown in Figures 8 and 9.

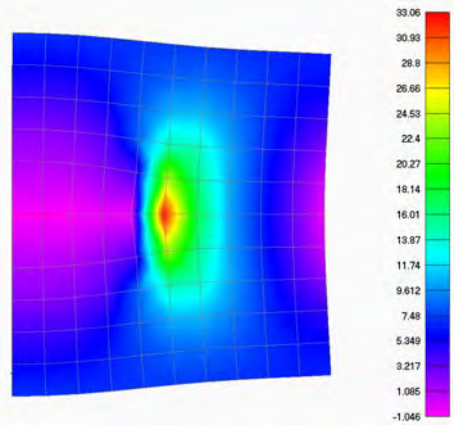


Fig. 8. *Normal stress field σ_{yy} (XFEM(H)), 10x10*

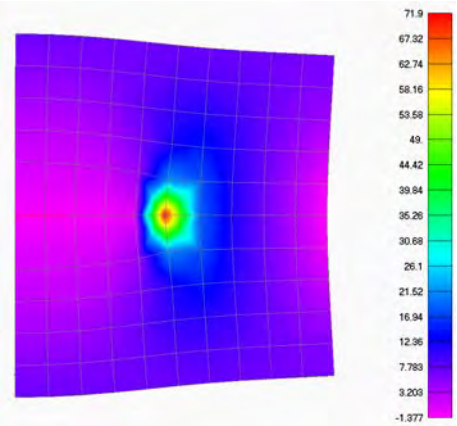


Fig. 9. *Normal stress field σ_{yy} (XFEM(H+NT)), 10x10*

It is obvious, in case of enrichment with NT, that there is rapid growth of stress when asymptotically approaching the crack tip, which is consistent with analytical solution.

4. Conclusions

The accuracy of the XFEM method was analyzed in this paper. Analysis of the effect of mesh density and the types of enrichment in the XFEM on the accuracy of the numerical results is done by comparing them with empirical and analytical values.

From the obtained results, it can be concluded that the size of the integration domain should be considered because it can happen that a small integration domain includes only a small number of nodes. Increasing mesh density reduces the error and enrichment with NT functions give a more accurate approximation of the stress field and better agreement with the analytical results. It was seen that calculation results have a good agreement in the area around the crack tip for all sizes of mesh. All calculations show the character of the rapid growth of stress when asymptotically approaching top of the crack. There is a good agreement between numerical and analytical results for the displacements of nodes in x and y directions depending on the vertical distance from the crack tip.

Acknowledgements

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