EVALUATION OF MEASUREMENT UNCERTAINTY IN TESTS REQUIRING NON-LINEAR REGRESSION

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Abstract: This work is concerned with the evaluation of measurement uncertainty arising from the use of non-linear regression in testing. While modern standards require a measured value to be accompanied by a statement of its quality in the form of an associated uncertainty, for many applications existing work procedures make no reference to uncertainty. Such procedures should therefore be updated to take account of the requirement to provide uncertainty information. This paper considers an application that involves the use of non-linear regression and for which uncertainty evaluation does not constitute part of the current work procedure. An updated procedure is proposed and example results are presented.

Keywords: Uncertainty, non-linear regression, modelling, least squares adjustment.

1. INTRODUCTION

Nowadays, according to the international standards regulating work in laboratories [1] and the presentation of results in testing [2][3], virtually all fields of science require that a measured value be accompanied by an associated uncertainty. For many applications, however, the standards that specify work procedures were written prior to the publication of the "Guide to the expression of uncertainty in measurement" (GUM) [4] and make no reference to the evaluation or presentation of uncertainties. Despite the modern requirement to provide uncertainty information, many engineering testing laboratories still do not provide such information as part of their measurement results.

This paper considers an application, concerned with soil compaction, for which the work procedure [5] does not describe uncertainty evaluation, and discusses how the procedure may be adapted to account for measurement uncertainty. The application requires a function to be fitted to measured data followed by the determination of quantities dependent on the parameters of the fitted function.

The paper is organized as follows. Section 2 provides background to the soil compaction application and describes the work procedure currently in place. An updated procedure that takes account of uncertainties associated with the measured data and allows uncertainties associated with the outputs of the procedure to be evaluated, is proposed in Section 3. Section 4 contains example results that illustrate the implementation of the updated procedure while conclusions are drawn in Section 5.

2. SOIL COMPACTION TESTS

Compaction tests are of great practical use, namely in the determination of reference parameters (e.g., optimum moisture content and maximum dry density) normally used in the control of results obtained during compaction works *in situ*, e.g., on embankments and earthen dams.

Compaction of soil is a mechanical process, consisting of fast and repeated application of a vertical load. The solid particles become more closely packed together, thus increasing the dry density of the soil and leading to a larger area of contact among the solid particles and an increased capacity to withstand loads. The dry density that can be achieved depends on the amount of moisture present in the soil and, for a given degree of compaction, there is an optimum moisture content at which the dry density reaches a maximum value.

Testing involves the compaction, in layers, of a soil sample within a compaction mould of specified dimensions, by means of an applied number of blows from a rammer of normalized weight, dropping from a normalized height, over the surface of each of the soil layers in the mould. This procedure is repeated for different amounts of moisture present in the soil in order to obtain six specimens.

The moisture content value W_i for the ith specimen, expressed as a percentage, is given by the average of the moisture content values determined for two samples of the specimen. That is,

$$W_i = \frac{W_{i,1} + W_{i,2}}{2}, \tag{1}$$

with

$$W_{i,j} = \frac{100\left(m_{2,i,j} - m_{3,i,j}\right)}{m_{3,i,j} - m_{1,i,j}}, \tag{2}$$

where $m_{1,i,j}$ is the mass of the baseplate used for the jth sample of the ith specimen, $m_{2,i,j}$ is the combined mass of the baseplate and the jth bulk compacted sample of the ith

specimen and $m_{3,i,j}$ is the combined mass of the baseplate and the jth dry compacted sample of the ith specimen.

The dry density $\gamma_{s,i}$ for the ith specimen is given by

$$\gamma_{s,i} = \frac{100\left(P_{T,i} - P_{M,i}\right)}{V_i\left(W_i + 100\right)}, \tag{3}$$

where $P_{M,i}$ is the mass of the compaction mould used for the ith specimen, $P_{T,i}$ is the combined mass of the compaction mould and the ith bulk compacted specimen and V_i is the volume of the ith bulk compacted specimen.

According to the work procedure in the standard concerned with compaction-related tests [5], for each compacted specimen, the measured values of dry density (denoted by γ_s) are plotted against the corresponding measured values of moisture content (W). A curve of best fit (compaction curve) to the six points is drawn and, using this curve, estimates of the maximum dry density $\gamma_{s(max)}$ and the corresponding optimum moisture content $W_{(opt)}$, i.e., the value of moisture content at which the maximum value of dry density occurs, are determined.

Clearly the nature of the best-fit function will influence the estimates of maximum dry density and optimum moisture content of the soil. However, the procedure provides no guidance regarding either the type of function or the criteria to use to determine the best-fit compaction curve. The compaction curve is generally determined using a simple Microsoft Excel spreadsheet application or, less frequently, drawn by hand and therefore it is conceivable for two individuals analysing the same measurement data to return different estimates of the reference parameters.

The main steps in the current procedure may therefore be summarized as follows:
1. Collect measurement data.
2. Calculate measured values of moisture content and dry density.
3. Plot measured values and determine best-fit compaction curve.
4. Determine estimates of reference parameters.

3. PROPOSED WORK PROCEDURE ACCOUNTING FOR UNCERTAINTY

As mentioned in Section 1, the current work procedure [5] does not describe how uncertainties associated with the measurement data can be used to determine uncertainties associated with the estimates of the reference parameters. A proposed approach that accounts for uncertainties is outlined below.

Throughout this section, vectors and matrices are used, allowing calculations to be written compactly using matrix-vector or matrix-matrix operations.

A vector of quantities is represented by a bold upper-case letter, e.g., $\mathbf{Z}$, while an estimate or value of that set of quantities is represented by the corresponding bold lower-case letter, e.g., $\mathbf{z}$.

Procedure: From equations (1)–(3), the moisture content W_i may be expressed as a function of $m_{1,i,1}$, $m_{1,i,2}$, $m_{2,i,1}$, $m_{2,i,2}$, $m_{3,i,1}$ and $m_{3,i,2}$, while the dry density $\gamma_{s,i}$ is a function of $m_{1,i,1}$, $m_{1,i,2}$, $m_{2,i,1}$, $m_{2,i,2}$, $m_{3,i,1}$, $m_{3,i,2}$, $P_{M,i}$, $P_{T,i}$ and V_i. Given measured values of these quantities, together with their associated uncertainties, corresponding values of W_i and $\gamma_{s,i}$ and their associated uncertainties can be determined. The dependence of both W_i and $\gamma_{s,i}$ on $m_{1,i,1}$, $m_{1,i,2}$, $m_{2,i,1}$, $m_{2,i,2}$, $m_{3,i,1}$ and $m_{3,i,2}$ means that they are correlated and a covariance term may be calculated to quantify the effect of this correlation. Additionally, in practice, the same compaction mould may be used more than once during a test, causing some of the quantities $\gamma_{s,i}$ to be correlated with each other.

Let
$$\mathbf{A} = (m_{1,1,1}, \ldots, m_{3,6,2}, P_{T,1}, \ldots, P_{T,6}, P_{M,1}, \ldots, P_{M,6}, V_1, \ldots, V_6)^T \tag{4}$$

be the vector of dimension 54×1 of quantities to be measured and

$$\mathbf{Z} = (W_1, \ldots, W_6, \gamma_{s,1}, \ldots, \gamma_{s,6})^T$$

be the vector of dimension 12×1 of moisture contents and dry densities.

Then
$$\mathbf{Z} = \mathbf{f}(\mathbf{A}) \equiv (f_1(\mathbf{A}), \ldots, f_{12}(\mathbf{A}))^T,$$

i.e., each element of $\mathbf{Z}$ is expressed as a function of $\mathbf{A}$.

Given measurement data $\mathbf{a}$, corresponding measured values $\mathbf{z}$ are given by
$$\mathbf{z} = \mathbf{f}(\mathbf{a}).$$

The covariance matrix $V_{\mathbf{z}}$ of dimension 12×12 associated with $\mathbf{z}$ is determined as follows. Construct the covariance matrix $V_{\mathbf{a}}$ of dimension 54×54 associated with $\mathbf{a}$:

$$V_{\mathbf{a}} = \begin{bmatrix} u^2(a_1) & u(a_1, a_2) & \cdots & u(a_1, a_{54}) \\ u(a_2, a_1) & u^2(a_2) & \cdots & u(a_2, a_{54}) \\ \vdots & \vdots & \ddots & \vdots \\ u(a_{54}, a_1) & u(a_{54}, a_2) & \cdots & u^2(a_{54}) \end{bmatrix},$$

where $u(a_p)$ is the standard uncertainty associated with a_p and $u(a_p, a_q)$ is the covariance associated with a_p and a_q.

Construct the Jacobian matrix $J_{\mathbf{a}}$ of dimension 12×54 of partial derivatives of the functions f_i with respect to the elements of $\mathbf{A}$, evaluated at $\mathbf{a}$:

$$J_{\mathbf{a}} = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial A_1}\right|_{\mathbf{A}=\mathbf{a}} & \cdots & \left.\dfrac{\partial f_1}{\partial A_{54}}\right|_{\mathbf{A}=\mathbf{a}} \\ \vdots & \ddots & \vdots \\ \left.\dfrac{\partial f_{12}}{\partial A_1}\right|_{\mathbf{A}=\mathbf{a}} & \cdots & \left.\dfrac{\partial f_{12}}{\partial A_{54}}\right|_{\mathbf{A}=\mathbf{a}} \end{bmatrix}.$$

Since each element of $\mathbf{Z}$ depends only on a subset of at most nine of the elements of $\mathbf{A}$, most of the partial derivatives will be zero. Analytical expressions for the non-zero partial derivatives may be obtained straightforwardly.

The covariance matrix $V_{\mathbf{z}}$ associated with $\mathbf{z}$ is then given by

$$V_{\mathbf{z}} = J_{\mathbf{a}} V_{\mathbf{a}} J_{\mathbf{a}}^{\mathrm{T}}. \tag{5}$$

The compaction curve models the dry density γ_{s} as a function of the moisture content W and a set of n_b (< 6) parameters $\mathbf{B}$ that are dependent on the choice of model function:

$$\gamma_{\mathrm{s}} = g(W, \mathbf{B}).$$

The approach proposed in this paper takes into account the uncertainty and covariance information stored in the covariance matrix $V_{\mathbf{z}}$ and requires the solution of a *generalized Gauss-Markov regression problem* (GGMR) [6] to determine the compaction curve.

Let
$$\mathbf{D} = \left(B_1, \ldots, B_{n_b}, W_1, \ldots, W_6\right)^{\mathrm{T}}.$$

It is required to determine estimates $\mathbf{d}$ of $\mathbf{D}$ such that the generalized sum of squares of residuals

$$(\mathbf{z} - \mathbf{c}(\mathbf{D}))^{\mathrm{T}} V_{\mathbf{z}}^{-1} (\mathbf{z} - \mathbf{c}(\mathbf{D})), \tag{6}$$

where
$$\mathbf{c}(\mathbf{D}) = \left(W_1, \ldots, W_6, g(W_1, \mathbf{B}), \ldots, g(W_6, \mathbf{B})\right)^{\mathrm{T}}$$

is a vector function of $\mathbf{D}$, is minimized.

Expression (6) can be rewritten as
$$\left(L_{\mathbf{z}}^{-1} \mathbf{h}(\mathbf{D})\right)^{\mathrm{T}} L_{\mathbf{z}}^{-1} \mathbf{h}(\mathbf{D}),$$

where $L_{\mathbf{z}}$ is the Cholesky factor [7] of dimension 12×12 of $V_{\mathbf{z}}$, i.e., the lower-triangular matrix that satisfies

$$V_{\mathbf{z}} = L_{\mathbf{z}} L_{\mathbf{z}}^{\mathrm{T}},$$

and
$$\mathbf{h}(\mathbf{D}) \equiv \left(h_1(\mathbf{D}), \ldots, h_{12}(\mathbf{D})\right)^{\mathrm{T}} = \mathbf{z} - \mathbf{c}(\mathbf{D}).$$

One approach to solving this optimization problem is the Gauss-Newton algorithm as follows:

I. Calculate the Cholesky factorization $L_{\mathbf{z}}$ of $V_{\mathbf{z}}$.
II. Determine initial estimates $\mathbf{d}_1$ of the parameters to be fitted.
III. Set $l = 1$.
IV. Evaluate the function $\mathbf{h}(\mathbf{d}_l)$.
V. Construct the Jacobian matrix $J_{\mathbf{d}}$ of dimension $12 \times (n_b + 6)$ of partial derivatives of the functions h_k with respect to the elements of $\mathbf{D}$, evaluated at $\mathbf{d}_l$:

$$J_{\mathbf{d}} = \begin{bmatrix} \left.\dfrac{\partial h_1}{\partial D_1}\right|_{\mathbf{D}=\mathbf{d}_l} & \cdots & \left.\dfrac{\partial h_1}{\partial D_{n_b+6}}\right|_{\mathbf{D}=\mathbf{d}_l} \\ \vdots & \ddots & \vdots \\ \left.\dfrac{\partial h_{12}}{\partial D_1}\right|_{\mathbf{D}=\mathbf{d}_l} & \cdots & \left.\dfrac{\partial h_{12}}{\partial D_{n_b+6}}\right|_{\mathbf{D}=\mathbf{d}_l} \end{bmatrix}.$$

VI. Solve for $\mathbf{e}_l$ the system of equations

$$\left(\left[L_{\mathbf{z}}^{-1} J_{\mathbf{d}}\right]^{\mathrm{T}} L_{\mathbf{z}}^{-1} J_{\mathbf{d}}\right) \mathbf{e}_l = -\left(L_{\mathbf{z}}^{-1} J_{\mathbf{d}}\right)^{\mathrm{T}} \mathbf{h}(\mathbf{d}_l).$$

VII. Set
$$\mathbf{d}_{l+1} = \mathbf{d}_l + \mathbf{e}_l.$$

VIII. Set $l := l + 1$.
IX. Repeat steps IV to VIII until convergence has been achieved, e.g.,
$$|\mathbf{e}_l| < \rho,$$
where ρ is a suitable tolerance value. Set $\mathbf{d} = \mathbf{d}_l$.
X. Calculate the covariance matrix $V_{\mathbf{d}}$ associated with $\mathbf{d}$:
$$V_{\mathbf{d}} = \left(\left[L_{\mathbf{z}}^{-1} J_{\mathbf{d}}\right]^{\mathrm{T}} L_{\mathbf{z}}^{-1} J_{\mathbf{d}}\right)^{-1}.$$
XI. Estimates $\mathbf{b}$ of the compaction curve parameters are given by the first n_b elements of $\mathbf{d}$. The associated covariance matrix $V_{\mathbf{b}}$ is the $n_b \times n_b$ upper-left block of $V_{\mathbf{d}}$.

In step VI, $\mathbf{e}_l$ can be calculated using a generalized QR approach [7].

Testing the consistency of the measured data with the fitted model can provide confidence that the choice of model is appropriate. One such check is to compare the value of expression (6), evaluated at $\mathbf{D} = \mathbf{d}$, and denoted by χ_{obs}^2, with the 95th percentile of the χ_{ν}^2 probability distribution, with $\nu = 6 - n_b$. If χ_{obs}^2 exceeds the 95th percentile of χ_{ν}^2, then the test for consistency is deemed to have failed.

If the consistency check fails, it is appropriate to check that the uncertainty information provided is correct, both in terms of that associated with the measurement data $\mathbf{a}$ and that associated with the measured values $\mathbf{z}$. If the uncertainty information is correct, consideration should be given to using a different model for the compaction curve.

Let
$$\mathbf{P} = \left(W_{\mathrm{opt}}, \gamma_{\mathrm{s(max)}}\right)^{\mathrm{T}}$$
be the vector of dimension 2×1 of reference parameters.

This paper considers only the case where the reference parameters can be expressed as explicit functions of the model parameters, i.e.,

$$\mathbf{P} = \boldsymbol{\varphi}(\mathbf{B}) \equiv \left(\phi_1(\mathbf{B}), \phi_2(\mathbf{B})\right)^{\mathrm{T}}.$$

For cases where the reference parameters can be expressed only as implicit functions of the model parameters see, e.g., [8].

Two approaches to uncertainty evaluation are considered: propagation of uncertainties [4, 8] and propagation of distributions [8, 9].

Propagation of uncertainties (PU): This approach requires a model that expresses the reference parameters in terms of the parameters of the compaction curve. The (vector) model function is linearized and the uncertainties and covariances associated with the estimates of the

compaction curve parameters are propagated through the linearized model to provide uncertainties associated with the estimates of the reference parameters.

Estimates **p** of the reference parameters are given by

$$\mathbf{p} = \left(\phi_1(\mathbf{b}), \phi_2(\mathbf{b})\right)^{\mathrm{T}}.$$

The covariance matrix $V_{\mathbf{p}}$ associated with the estimates of the reference parameters is given by

$$V_{\mathbf{p}} = J_{\mathbf{p}} V_{\mathbf{b}} J_{\mathbf{p}}^{\mathrm{T}},$$

where $J_{\mathbf{p}}$ is the Jacobian matrix of dimension $2 \times n_b$ of partial derivatives of the functions ϕ_1 and ϕ_2 with respect to the elements of **B**, evaluated at **b**:

$$J_{\mathbf{p}} = \begin{bmatrix} \dfrac{\partial \phi_1}{\partial B_1}\bigg|_{\mathbf{B}=\mathbf{b}} & \cdots & \dfrac{\partial \phi_1}{\partial B_{n_b}}\bigg|_{\mathbf{B}=\mathbf{b}} \\ \dfrac{\partial \phi_2}{\partial B_1}\bigg|_{\mathbf{B}=\mathbf{b}} & \cdots & \dfrac{\partial \phi_2}{\partial B_{n_b}}\bigg|_{\mathbf{B}=\mathbf{b}} \end{bmatrix}.$$

The bivariate Gaussian distribution $\mathrm{N}(\mathbf{p},\ V_{\mathbf{p}})$ is then assigned to the reference parameters.

Propagation of distributions (PD): This approach implements the propagation of distributions through the model for the reference parameters, making no linear approximation as in the GUM uncertainty framework:

I. Select the number M of Monte Carlo trials.
II. Draw samples $\mathbf{b}_q$, $q = 1,\ldots,M$, from the multivariate Gaussian distribution $\mathrm{N}(\mathbf{b}, V_{\mathbf{b}})$.
III. Calculate corresponding reference parameter values $\mathbf{p}_q$, $q = 1,\ldots,M$.
IV. Calculate statistics (mean, standard deviation and covariance) for the aggregated reference parameter values. Scaled histograms of the reference parameter values may be generated to provide approximations to their probability distributions.

The estimates **d** are a mildly non-linear function of **z**, and the covariance matrix $V_{\mathbf{d}}$ associated with **d** can be determined by the PU approach. However, the non-linearity of the models for the reference parameters (see, e.g., Section 4) suggests that it may be appropriate to use PD for uncertainty evaluation. For a given model, for particular measurement data, PU may return acceptable answers for some, but not necessarily all, quantities. However, there is no guarantee that, for the same model, this approach will return suitable values for other measurement data. The use of PD is therefore encouraged.

The main steps in the proposed procedure may therefore be summarized as follows:
1. Select model $\gamma_s = g(W, \mathbf{B})$ for compaction curve.
2. Collect measurement data **a**.
3. Calculate measured values **z**.
4. Construct covariance matrix $V_{\mathbf{a}}$.
5. Construct Jacobian matrix $J_{\mathbf{a}}$.
6. Calculate covariance matrix $V_{\mathbf{z}}$.

7. Implement generalized Gauss-Markov regression fit to measured values to obtain estimates **b** of model parameters and associated covariance matrix $V_{\mathbf{b}}$.
8. Check for consistency of measured values with fitted model.
9. If the consistency check in step 8 fails, return to step 1. Otherwise, determine estimates **p** of reference parameters and associated uncertainty information.

4. EXAMPLE RESULTS

Table 1 summarizes the measurements made for a single test. For this study, the same cylindrical compaction mould was used for all specimens, denoted in the table by single values P_{M} for the mass of the mould, and d and h for, respectively, the diameter and height of the mould.

The same balance was used to measure the masses $m_{1,i,1}$, $m_{1,i,2}$, $m_{2,i,1}$, $m_{2,i,2}$, $m_{3,i,1}$, $m_{3,i,2}$, $i = 1,\ldots,6$. For any mass m measured by this balance, the rectangular probability distribution with limits $\hat{m} \pm 0.001\hat{m}$ is assigned to m, meaning that the standard uncertainty associated with the estimate $\hat{m}$ of m is $0.001\hat{m}/\sqrt{3}$.

A second balance was used to measure masses $P_{\mathrm{T},i}$, $i = 1,\ldots,6$, and P_{M}. For any mass P measured by this balance, the rectangular probability distribution with limits $\hat{P} \pm 1\,\mathrm{g}$ is assigned to P, meaning that the standard uncertainty associated with the estimate $\hat{P}$ of P is $1/\sqrt{3}$ g.

Rectangular probability distributions with limits $\hat{d} \pm 0.04\,\mathrm{cm}$ and $\hat{h} \pm 0.05\,\mathrm{cm}$ are assigned to d and h, respectively, meaning that the standard uncertainties associated with the estimates $\hat{d}$ of d and $\hat{h}$ of h are $0.04/\sqrt{3}$ cm and $0.05/\sqrt{3}$ cm.

The estimate $\hat{V}$ of the volume of the mould and its associated standard uncertainty $u(\hat{V})$ are determined using the formulae

$$\hat{V} = \frac{\pi(\hat{d})^2 \hat{h}}{4}, \quad u^2(\hat{V}) = \left(\frac{\pi \hat{d}\hat{h}}{2}\right)^2 u^2(\hat{d}) + \left(\frac{\pi(\hat{d})^2}{4}\right)^2 u^2(\hat{h}).$$

The measured values of W_i and $\gamma_{s,i}$ are determined using equations (1)–(3) and are plotted in Figure 1.

The repeated use of the same compaction mould means that the vector of parameters **A** in expression (4) reduces to one of dimension 44×1. Consequently, the Jacobian matrix $J_{\mathbf{a}}$ and covariance matrix $V_{\mathbf{a}}$ are of dimension 12×44 and 44×44, respectively. The covariance matrix $V_{\mathbf{z}}$ associated with the values of moisture content and dry density is calculated using expression (5).

A cubic polynomial compaction curve was fitted to the measured values. For numerical stability, Chebyshev

polynomials [10] are used rather than a monomial representation:

$$g(W, \mathbf{B}) = \tfrac{1}{2} B_1 T_0\big(\hat{W}\big) + B_2 T_1\big(\hat{W}\big) + B_3 T_2\big(\hat{W}\big) + B_4 T_3\big(\hat{W}\big),$$

where

$$\mathbf{B} = \big(B_1, B_2, B_3, B_4\big)^{\mathrm{T}}, \ n_b = 4,$$
$$T_0\big(X\big) = 1, \ T_1\big(X\big) = X,$$
$$T_j\big(X\big) = 2X T_{j-1}\big(X\big) - T_{j-2}\big(X\big), \ j \geq 2,$$

and $\hat{W}$ is the normalized variable given by

$$\hat{W} = \frac{2W - \big(W_{\max} + W_{\min}\big)}{W_{\max} - W_{\min}},$$

where $W_{\min}$ and $W_{\max}$ are chosen so that $W_{\max} > W_{\min}$ and the interval $[W_{\min}, W_{\max}]$ contains the measured values.

The consistency check returns a result of $\chi^2_{\mathrm{obs}} = 4.18$, which compares favourably with the 95th percentile of the χ^2_ν probability distribution, with $\nu = 6 - 4 = 2$, namely 5.99.

For a cubic polynomial compaction curve, the estimate of the optimum moisture content is given by a zero of the derivative of the curve, which is a quadratic function, and can therefore be expressed as an explicit function of the compaction curve parameters. The corresponding maximum dry density is obtained by evaluating the compaction curve for the optimum moisture content and is therefore also expressed as an explicit function of the compaction curve parameters.

Figure 1 shows the fitted compaction curve. Both the PU and PD (with $M = 100\,000$) approaches to uncertainty evaluation were implemented for the reference parameters. Figures 2 and 3 show the probability distributions obtained using the two approaches for, respectively, the optimum moisture content and the maximum dry density. Table 2 shows the estimates of the reference parameters, their associated standard uncertainties and symmetric 95 % confidence intervals obtained using the two approaches. (In practice, results are often reported using fewer digits of precision than are used in this paper.)

For both reference parameters, the distributions obtained using PU and PD are very similar, and the estimates, standard uncertainties and confidence intervals are almost the same (to the reported numerical precision).

The evaluated uncertainties arise from the assumption that the cubic model represents the true physical response. However, this model is derived empirically and the true physical response may be somewhat different, thus there is an uncertainty contribution associated with the choice of model that is not accounted for by the methodology.

5. CONCLUSIONS

This paper proposes an updated procedure for a soil compaction application, taking measurement uncertainty into account. The existing procedure (a) does not consider uncertainty, and (b) is such that the choice of compaction curve is highly dependent on the individual processing the measurement data. An important property of the proposed procedure is that, having decided on the nature of the compaction curve model to be used, for given measurement data and associated uncertainty information, the compaction curve is uniquely defined, and therefore so are the corresponding reference parameters.

The proposed procedure allows uncertainty information associated with the reference parameters of the compaction curve to be determined. Two approaches to uncertainty evaluation are described, although it is recommended that the propagation of distributions be implemented.

The results of applying the proposed procedure to example measurement data are provided.

6. ACKNOWLEDGEMENTS

7. REFERENCES

[1] ISO/IEC 17025:2005, General requirements for the competence of testing and calibration laboratories.

[2] EA 4/16, EA guidelines on the expression of uncertainty in quantitative testing, 2003.

[3] ILAC-P14:12/2010, ILAC policy for uncertainty in calibration.

[4] BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP and OIML, Guide to the expression of uncertainty in measurement. Joint Committee for Guides in Metrology, Sèvres, France, 2008.

[5] BS 1377-4:1990, Methods of tests for soils for civil engineering purposes. Compaction-related tests.

[6] M. G. Cox, A. B. Forbes, P. M. Harris and I. M. Smith, The classification and solution of regression problems for calibration. NPL Report CMSC 24/03, 2003.

[7] G. H. Golub and C. F. Van Loan, Matrix Computations. John Hopkins University Press, 1996.

[8] BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP and OIML, Evaluation of measurement data – Supplement 2 to the "Guide to the expression of uncertainty in measurement" – Extension to any number of output quantities. Joint Committee for Guides in Metrology, Sèvres, France, 2010.

[9] BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP and OIML, Evaluation of measurement data – Supplement 1 to the "Guide to the expression of uncertainty in measurement" – Propagation of distributions using a Monte Carlo method. Joint Committee for Guides in Metrology, Sèvres, France, 2008.

[10] L. Fox and I. B. Parker, Chebyshev polynomials in numerical analysis. Oxford University Press, 1968.

Table 1 – Example measurement data

		i					
		1	2	3	4	5	6
Quantity	$m_{1,i,1}$/g	108.441	107.384	103.831	105.690	109.667	106.441
	$m_{1,i,2}$/g	105.089	108.657	104.335	105.876	107.062	103.510
	$m_{2,i,1}$/g	341.123	365.755	392.359	421.277	446.173	442.123
	$m_{2,i,2}$/g	373.537	411.952	372.474	421.923	390.125	392.415
	$m_{3,i,1}$/g	297.200	313.399	330.254	348.608	363.828	357.961
	$m_{3,i,2}$/g	322.810	349.886	314.364	348.682	321.106	320.213
	$P_{T,i}$/g	3274	3372	3465	3572	3592	3574
	P_M/g	1836					
	d/cm	10.19					
	h/cm	11.71					

Table 2 – Estimates of reference parameters, associated standard uncertainties and symmetric 95 % confidence intervals obtained using PU and PD

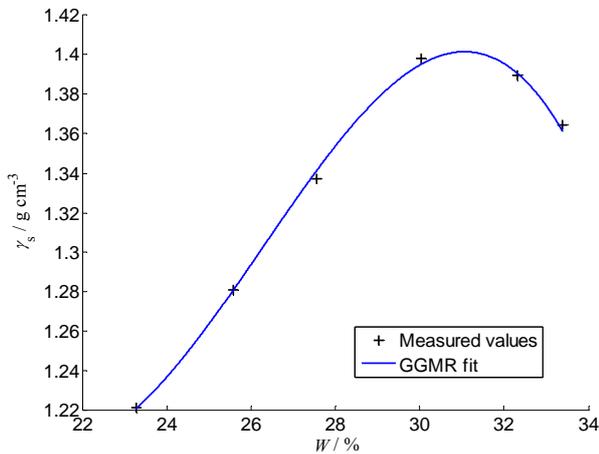|	PU	PD
$W_{(opt)}$ /%	31.07	31.07
$u(W_{(opt)})$ /%	0.05	0.06
Coverage interval for $W_{(opt)}$ /%	[30.96, 31.18]	[30.96, 31.18]
$\gamma_{s(max)}$ /gcm^{-3}	1.401	1.402
$u(\gamma_{s(max)})$ /gcm^{-3}	0.007	0.007
Coverage interval for $\gamma_{s(max)}$ /gcm^{-3}	[1.387, 1.416]	[1.387, 1.416]

Figure 2 –Probability distributions for optimum moisture content
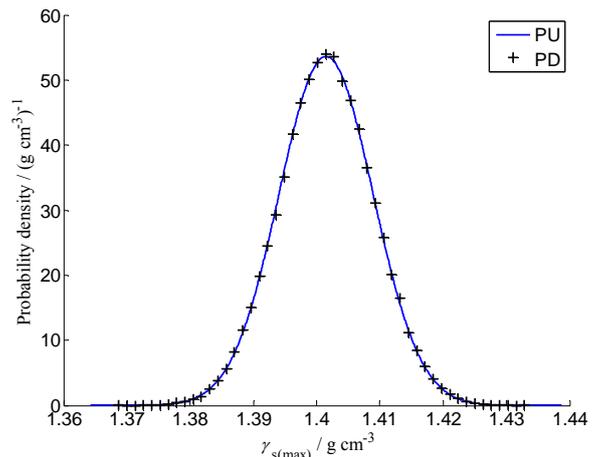

Figure 1 – Measurement data and fitted compaction curve

Figure 3 – Probability distributions for maximum dry density