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STATE-SPACE BASED ADAPTATION OF THE ISO GUM TO TIME-DEPENDENT UNCERTAINTIES OF DYNAMICALLY MEASURED QUANTITIES

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***Abstract*—The well-established ISO GUM method for evaluating measurement uncertainties refers to measurements where the properties of all involved quantities are assumed to be constant. In many technical applications, however, a time-dependent uncertainty information that can be attributed to values measured and filtered by dynamic systems is more and more needed, e.g. for closed-loop control. This contribution extends the existing ISO GUM method consistently to time-dependent quantities of dynamic systems in a straightforward way. For this purpose, state-space models are used, as this approach results in a method that is very close to the original ISO GUM. The method is demonstrated by using the example of a low-pass filter.**

***Index Terms*—ISO GUM, dynamic, uncertainty, state-space**

I. Introduction

The present form of the well-established ISO GUM method can only be applied to static measurement systems and is not suitable for dynamic measurement tasks such as the position measurement of moving objects [1]. The processing of values by dynamic systems is not aimed by the ISO GUM method. Thus no standard compliant uncertainty information can be attributed to the system's output values even if the uncertainties of the input values are given. Although methods for analysing and handling the behaviour of dynamic systems are well-researched in signal processing and control engineering, the knowledge transfer to standardised metrological methods is still a challenging task [2]. The ISO GUM framework has to be enhanced, if the standardised evaluation of dynamic measurement systems resulting in so-called *dynamic uncertainties* [3] is required. In [4] state-space models are considered as promising means of description for the measurement dynamic, due to their advantages in a later implementation. Additionally state-space models can be analysed with the known methods from systems theory in a mathematically elegant way.

In this paper, the GUM algorithm, as it is, is applied in a straightforward way to a linear state-space model. Thus estimates and their uncertainties of its output quantities are obtained in each time step. A second state-space model is deduced describing the uncertainty matrix's evolution over time that permits a stability analysis of the varying uncertainty matrix.

Firstly the existing ISO GUM method is briefly summarised in Section II. In Section III, the GUM method is extended to state-space models and the evolution of the uncertainty matrix over time is analysed. Section IV illustrates the developed method by considering a simple first order low-pass filter. Finally the results of the paper are concluded in Section V.

II. Summary of the ISO GUM method

This section summarises the ISO GUM method briefly including the new supplement 2 (*Extension to any number of output quantities* [5]). According to [6], the method can be divided into eight steps.

The task within Step I is to develop a measurement model that reflects the relations between all input and output quantities of the measurement in a sufficient way. Supplement 2 [5] extends the model for a single output to a general one

$$\boldsymbol{Y} = \mathbf{f}(\boldsymbol{X}) \tag{1}$$

with an arbitrary number of measurands, where $\mathbf{f}$ is a vector function that maps the vector of input quantities $\boldsymbol{X} \in \mathbb{R}^{\mathrm{R}}$ onto the vector of output quantities $\boldsymbol{Y} \in \mathbb{R}^{\mathrm{M}}$.

After the relevant input quantities X_r $(r = 1, ..., \mathrm{R})$ and the measurands of interest Y_m $(m = 1, ..., \mathrm{M})$ are identified, the most probable input values $\bar{\boldsymbol{x}}$ have to be determined in Step II.

Step III consists of the evaluation of the standard uncertainties $\mathrm{u}_{\bar{x}_r}$. The notation $\mathrm{u}_{\bar{x}_r}$ denotes that the uncertainty u is attributed to the estimate $\bar{x}_r$ of X_r. A standard uncertainty can be obtained by a statistical analysis of observations (evaluation method *Type A*). If no observations are possible or desired, the standard uncertainties are determined by means of metrologist's expertise, data sheets, or information from literature (evaluation method *Type B*).

The covariances $\mathrm{u}_{\bar{x}_r,\bar{x}_l}$ $(r, l = 1, ..., \mathrm{R})$ of each pair of input estimates have to be evaluated in Step IV. The squared standard uncertainties $\mathrm{u}^2_{\bar{x}_r}$ and the covariances $\mathrm{u}_{\bar{x}_r,\bar{x}_l}$ constitute the input uncertainty matrix $\mathbf{U}_{\bar{\boldsymbol{x}}}$ that can be reasonably attributed to the best input estimates $\bar{\boldsymbol{x}}$.

The best estimates $\bar{\boldsymbol{y}}$ of the output quantities $\boldsymbol{Y}$ are calculated by applying the best input estimates $\bar{\boldsymbol{x}}$ to the linearised measurement function $\mathbf{f}$ (Step V). Expanding the measurement function (1) as a Taylor series results in the law of linear uncertainty propagation

$$\mathbf{U}_{\bar{\boldsymbol{y}}} = \mathbf{J}\mathbf{U}_{\bar{\boldsymbol{x}}}\mathbf{J}^{\mathrm{T}}, \tag{2}$$

where $\mathbf{J}$ is the Jacobian of the measurement function (1) evaluated at the best estimates $\bar{\boldsymbol{x}}$ [5]. Equation (2) is used in Step VI to calculate the uncertainty matrix of the measurands' estimates.

If required, the coverage interval can be enlarged in Step VII by determining an expanded uncertainty

$$\mathrm{U}_{\bar{y}_m} = \mathrm{k}_{\mathrm{P},m}\mathrm{u}_{\bar{y}_m}.$$

The coverage factor $\mathrm{k}_{\mathrm{P},m}$ depends on the probability density function assumed for the measurands and on the requested probability P, which is typically 95%.

The ISO GUM considers a summarised report of all work carried out as an important, explicit Step VIII. The report has to contain the best estimates $\bar{y}_m$, their covariances $\mathrm{u}_{\bar{y}_m,\bar{y}_w}$ $(m, w = 1, ..., \mathrm{M})$, their expanded uncertainties $\mathrm{U}_{\bar{y}_m}$, the coverage factors $\mathrm{k}_{\mathrm{P},m}$, and every additional information that is necessary to reproduce the result.

III. Adaptation of the ISO GUM method to dynamic systems

Compared to the standard ISO GUM method, the Steps I, V, and VI have to be adapted as described in the following. In doing so, changes to the original method are reduced to a minimum for consistency reasons.

A. Step I: Modelling of the measurement

Standardised uncertainty evaluation for dynamic measurements requires the extension of the GUM method to the class of systems that have an internal state. Here the time-discrete state-space model

$$\begin{aligned} \boldsymbol{Z}(k+1) &= \mathbf{f}_{\boldsymbol{Z}}(\boldsymbol{Z}, \boldsymbol{X}) = \mathbf{A}\boldsymbol{Z}(k) + \mathbf{B}\boldsymbol{X}(k), \quad \boldsymbol{Z}(0) = \boldsymbol{Z}_0 \\ \boldsymbol{Y}(k) &= \mathbf{f}_{\boldsymbol{Y}}(\boldsymbol{Z}, \boldsymbol{X}) = \mathbf{C}\boldsymbol{Z}(k) + \mathbf{D}\boldsymbol{X}(k). \end{aligned} \tag{3}$$

is used for describing the dynamic measurement system. The state function $\mathbf{f}_{\boldsymbol{Z}}(\boldsymbol{Z}(k), \boldsymbol{X}(k))$ and the output function $\mathbf{f}_{\boldsymbol{Y}}(\boldsymbol{Z}(k), \boldsymbol{X}(k))$ are assumed to be linear. The system matrix $\mathbf{A} \in \mathbb{R}^{(\mathrm{N}\times\mathrm{N})}$, the input matrix $\mathbf{B} \in \mathbb{R}^{(\mathrm{N}\times\mathrm{R})}$, the output matrix $\mathbf{C} \in \mathbb{R}^{(\mathrm{M}\times\mathrm{N})}$, the feedthrough matrix $\mathbf{D} \in \mathbb{R}^{(\mathrm{M}\times\mathrm{R})}$ as well as the initial state $\boldsymbol{Z}_0$ represent the parameters of the state-space model to be determined. The vector $\boldsymbol{X} = \left(X_1, ..., X_{\mathrm{R}}\right)^{\mathrm{T}} \in \mathbb{R}^{\mathrm{R}}$ contains all input quantities, whose best estimates and uncertainties propagate through the dynamic system. Analogously $\boldsymbol{Z} = \left(Z_1, ..., Z_{\mathrm{N}}\right)^{\mathrm{T}} \in \mathbb{R}^{\mathrm{N}}$ denotes the state vector and $\boldsymbol{Y} = \left(Y_1, ..., Y_{\mathrm{M}}\right)^{\mathrm{T}} \in \mathbb{R}^{\mathrm{M}}$ the vector containing all output quantities.

B. Step V: Calculation of the best state and output estimates

The current best estimates $\bar{\boldsymbol{z}}(k)$ of the state quantities $\boldsymbol{Z}(k)$ and the best estimates $\bar{\boldsymbol{x}}(k)$ of the input quantities $\boldsymbol{X}(k)$ yield the subsequent best state estimates

$$\bar{\boldsymbol{z}}(k+1) = \mathbf{A}\bar{\boldsymbol{z}}(k) + \mathbf{B}\bar{\boldsymbol{x}}(k),$$

as (3) is a linear system. The best estimates of the outputs

$$\bar{\boldsymbol{y}}(k) = \mathbf{C}\bar{\boldsymbol{z}}(k) + \mathbf{D}\bar{\boldsymbol{x}}(k)$$

are calculated in an analogue way.

C. Step VI: Calculation of the state and output uncertainties

In addition to the best estimates of the states and outputs, their corresponding uncertainties and covariances are propagated through the dynamic system. The law of uncertainty propagation holds in each time step according to (2):

$$\mathbf{U}_{\bar{\boldsymbol{z}}}(k+1) = \mathbf{J}_{\boldsymbol{Z}}\mathbf{U}_{\bar{\boldsymbol{z}},\bar{\boldsymbol{x}}}(k)\mathbf{J}_{\boldsymbol{Z}}^{\mathrm{T}}, \tag{4}$$

where

$$\mathbf{U}_{\bar{\boldsymbol{z}},\bar{\boldsymbol{x}}}(k) = \begin{pmatrix} \mathbf{U}_{\bar{\boldsymbol{z}}}(k) & \mathbf{O} \\ \mathbf{O} & \mathbf{U}_{\bar{\boldsymbol{x}}}(k) \end{pmatrix}$$

denotes the uncertainty matrix of the input and state quantities and

$$\mathbf{J}_{\boldsymbol{Z}} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \end{pmatrix}$$

the Jacobian of $\mathbf{f}_{\boldsymbol{Z}}(\boldsymbol{Z}(k), \boldsymbol{X}(k))$. The output uncertainty matrix

$$\mathbf{U}_{\bar{\boldsymbol{y}}}(k) = \mathbf{J}_{\boldsymbol{Y}}\mathbf{U}_{\bar{\boldsymbol{z}},\bar{\boldsymbol{x}}}(k)\mathbf{J}_{\boldsymbol{Y}}^{\mathrm{T}}, \tag{5}$$

is calculated with the Jacobian

$$\mathbf{J}_{\boldsymbol{Y}} = \begin{pmatrix} \mathbf{C} & \mathbf{D} \end{pmatrix}$$

of the output function $\mathbf{f}_{\boldsymbol{Y}}(\boldsymbol{Z}(k), \boldsymbol{X}(k))$ in the same way. Remarkably the matrices $\mathbf{J}_{\boldsymbol{X}}$ and $\mathbf{J}_{\boldsymbol{Y}}$ are constant as linear systems are considered. An evaluation of the Jacobians at the current best estimates $\bar{\boldsymbol{z}}(k)$ and $\bar{\boldsymbol{x}}(k)$ in each time step is not required. This makes the calculation of the time-dependent uncertainties very efficient. Hence this approach is well suited for real-time applications, where embedded systems with limited computing power are used very often.

D. Analysis of the system behaviour

For the analysis of the system behaviour, the progress of the best estimates $\bar{\boldsymbol{z}}(k)$ and $\bar{\boldsymbol{y}}(k)$ as well as of the uncertainties and covariances are considered. The behaviour of the best output estimates is described by

$$\bar{\boldsymbol{y}}(k) = \mathbf{C}\mathbf{A}^{k}\bar{\boldsymbol{z}}(0) + \sum_{j=1}^{k-1}\mathbf{C}\mathbf{A}^{k-1-j}\mathbf{B}\bar{\boldsymbol{x}}(j) + \mathbf{D}\bar{\boldsymbol{x}}(k),$$

which is the solution of the difference equation system (3) (see [7]).

For an stability analysis of the system (4), (4) is rewritten as

$$\begin{aligned}\mathbf{U}_{\bar{\boldsymbol{z}}}(k+1) &= \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix} \mathbf{U}_{\bar{\boldsymbol{z}},\bar{\boldsymbol{x}}}(k) \begin{pmatrix}\mathbf{A} & \mathbf{B}\end{pmatrix}^{\mathrm{T}} \\ &= \mathbf{A}\mathbf{U}_{\bar{\boldsymbol{z}}}(k)\mathbf{A}^{\mathrm{T}} + \mathbf{B}\mathbf{U}_{\bar{\boldsymbol{x}}}(k)\mathbf{B}^{\mathrm{T}}.\end{aligned} \tag{6}$$

To transform (6) into the common state-space form analogue to (3), the vectorisation of a matrix and the Kronecker product according to [8] are introduced in the following.

Definition 1 (Vectorisation) Let $\mathbf{U} \in \mathbb{C}^{(\mathrm{N}\times\mathrm{N})}$ with the column vectors $\mathbf{u}_n \in \mathbb{C}^{(\mathrm{N}\times 1)}$ $(n = 1, ..., \mathrm{N})$. Then

$$\mathrm{vec}\left(\mathbf{U}\right) := \begin{pmatrix}\mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{\mathrm{N}}\end{pmatrix} \in \mathbb{C}^{(\mathrm{N}^2\times 1)}$$

is the vectorisation of $\mathbf{U}$. $\Diamond$

Corollary 1 is a self-evident, but important property of the vectorisation, which represents the principle of superposition.

Corollary 1 Let $\mathbf{A} \in \mathbb{C}^{(\alpha\times\beta)}$ and $\mathbf{B} \in \mathbb{C}^{(\alpha\times\beta)}$ with the column vectors $\mathbf{a}_1, ..., \mathbf{a}_\beta$ and $\mathbf{b}_1, ..., \mathbf{b}_\beta$ respectively. Then

$$\mathrm{vec}\left(\mathbf{A}+\mathbf{B}\right) = \mathrm{vec}\left(\mathbf{A}\right) + \mathrm{vec}\left(\mathbf{B}\right)$$

holds. $\square$

Definition 2 (Kronecker product) Let $\mathbf{A} \in \mathbb{C}^{(\mathrm{N}\times\mathrm{N})}$ and $\mathbf{B} \in \mathbb{C}^{(\mathrm{N}\times\mathrm{R})}$. Then

$$\Lambda = (\mathbf{A}\otimes\mathbf{B}) := \begin{pmatrix}\mathrm{a}_{11}\mathbf{B} & \cdots & \mathrm{a}_{1\mathrm{N}}\mathbf{B} \\ \vdots & \ddots & \vdots \\ \mathrm{a}_{\mathrm{N}1}\mathbf{B} & \cdots & \mathrm{a}_{\mathrm{NR}}\mathbf{B}\end{pmatrix} \in \mathbb{C}^{(\mathrm{N}^2\times\mathrm{NR})}$$

is called the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$. $\Diamond$

Theorem 1 Let $\mathbf{A} \in \mathbb{C}^{(\mathrm{N}\times\mathrm{N})}$ and $\mathbf{U} \in \mathbb{C}^{(\mathrm{N}\times\mathrm{N})}$. Then

$$\mathrm{vec}\left(\mathbf{A}\mathbf{U}\mathbf{A}^{\mathrm{T}}\right) = (\mathbf{A}\otimes\mathbf{A})\mathrm{vec}\left(\mathbf{U}\right)$$

holds.

The reader is invited to prove Theorem 1 by transforming the reshaped quadratic term into the definition of the Kronecker product (Definition 2).

Using the vectorisation of the uncertainty matrix (Definition 1), the Kronecker product (Definition 2) as well as Corollary 1 and Theorem 1, the state equation (6) can be rewritten as

$$\begin{aligned}\mathrm{vec}\left(\mathbf{U}_{\bar{\boldsymbol{z}}}\right)(k+1) = (\mathbf{A}\otimes\mathbf{A})&\mathrm{vec}\left(\mathbf{U}_{\bar{\boldsymbol{z}}}\right)(k) \\ +(\mathbf{B}\otimes\mathbf{B})&\mathrm{vec}\left(\mathbf{U}_{\bar{\boldsymbol{x}}}\right)(k).\end{aligned} \tag{7}$$

Equation (7) has the form of a linear state equation, analogue to the difference equation of (3), with a state vector $\mathrm{vec}\left(\mathbf{U}_{\bar{\boldsymbol{z}}}\right)(k)$ containing the squared state uncertainties and the covariances. This allows the application of the well-known methods for the analysis of linear state-space systems as demonstrated in the following.

E. Stability of the state uncertainty

System (7) describing the evolution of the uncertainty matrix can be asymptotically stable, marginally stable, or unstable. If the condition mentioned in Theorem 2 holds, the system (7) is asymptotically stable and the eigensolution of its state equation fades away. This means that an existing state uncertainty can even decrease as soon as previous input values with large uncertainties represented by the state are followed by input values with minor uncertainties.

Theorem 2 The system (7) is asymptotically stable, if for all eigenvalues λ_n $(n = 1, ..., \mathrm{N})$ of $\mathbf{A}$ holds: $|\lambda_n| < 1$.

Proof 1 Let $\mathbf{A} \in \mathbb{C}^{(\mathrm{N}\times\mathrm{N})}$ with the eigenvalues λ_n $(n = 1, ..., \mathrm{N})$. Then the eigenvalues of $\Lambda = (\mathbf{A}\otimes\mathbf{A}) \in \mathbb{C}^{(\mathrm{N}^2\times\mathrm{N}^2)}$ are

$$(\lambda_n\lambda_\eta)\ (n = 1, ..., \mathrm{N}\ ;\ \eta = 1, ..., \mathrm{N})$$

[8]. If $|\lambda_n| < 1\ \forall\ \lambda_n$ is satisfied, then all eigenvalues of Λ are within the unit circle of the complex plane, too. The system (7) is asymptotically stable. $\square$

The system (7) is marginally stable, if $\mathbf{A}$ is diagonalisable and $|\lambda_n| = 1$ holds [9]. In this case, the elements of the state uncertainty matrix increase, if further uncertainty is conveyed to the system by the input quantities. Existing state uncertainty cannot degenerate. In case of instability, the existing state uncertainty would exceed all limits, even if the uncertainties associated with $\bar{\boldsymbol{x}}$ were zero.

F. Equilibrium of the state uncertainty

For a constant input uncertainty $\widehat{\mathbf{U}}_{\bar{\boldsymbol{x}}}$, the state of an asymptotically stable system (7) converges against an equilibrium state. The state uncertainty vector $\mathrm{vec}\left(\mathbf{U}_{\bar{\boldsymbol{z}}}\right)(k)$ represents an equilibrium state, if

$$\mathrm{vec}\left(\mathbf{U}_{\bar{\boldsymbol{z}}}\right)(k) = (\mathbf{A}\otimes\mathbf{A})\mathrm{vec}\left(\mathbf{U}_{\bar{\boldsymbol{z}}}\right)(k) + (\mathbf{B}\otimes\mathbf{B})\mathrm{vec}\left(\widehat{\mathbf{U}}_{\bar{\boldsymbol{x}}}\right) \tag{8}$$

holds. The equilibrium state uncertainty

$$\mathrm{vec}\left(\widehat{\mathbf{U}}_{\bar{\boldsymbol{z}}}\right) = (\mathbf{I}-(\mathbf{A}\otimes\mathbf{A}))^{-1}(\mathbf{B}\otimes\mathbf{B})\mathrm{vec}\left(\widehat{\mathbf{U}}_{\bar{\boldsymbol{x}}}\right) \tag{9}$$

is calculated by solving (8), where $\mathbf{I}$ is the identity matrix. The corresponding state uncertainty matrix $\widehat{\mathbf{U}}_{\bar{\boldsymbol{z}}}$ is given by writing the vector elements of $\mathrm{vec}\left(\widehat{\mathbf{U}}_{\bar{\boldsymbol{z}}}\right)$ in an $(\mathrm{N}\times\mathrm{N})$ matrix column-wise. The corresponding output uncertainty matrix is described by (5).

IV. Illustration Example

As a demonstration example a PT_1 system representing a low-pass filter is considered that is defined by the continuous state-space model

$$\begin{aligned}&\dot{Z}(t) = -\frac{1}{\mathrm{T}_1}Z(t) + \frac{1}{\mathrm{T}_1}X(t), \quad Z(0) = Z_0 \\ &Y(t) = \mathrm{k_s}Z(t),\end{aligned}$$

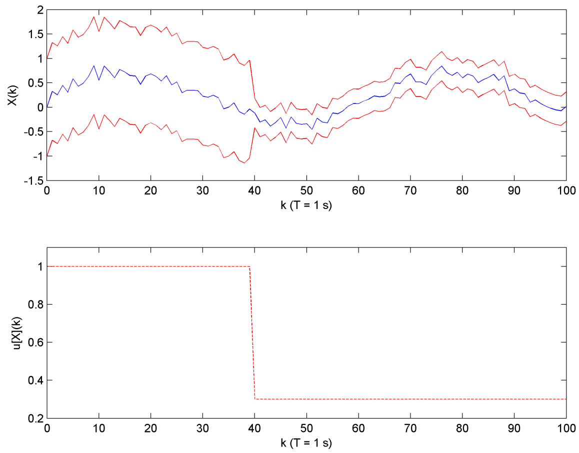

Fig. 1. Trajectory of the input quantity to be filtered and its uncertainty

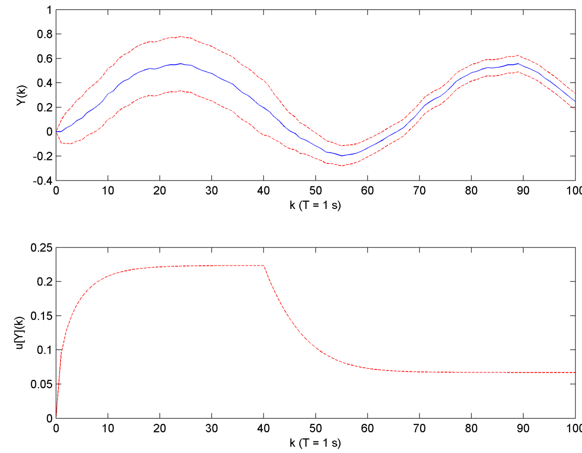

Fig. 2. Trajectory of the output quantity and its uncertainty

which is parameterised with the time constant T_1 and the static gain $\mathrm{k_s}$. The sampling with the period T yields the parameters

$$\mathbf{A} = e^{-\frac{\mathrm{T}}{\mathrm{T}_1}} = \alpha$$

and

$$\mathbf{B} = \int_0^t e^{-\frac{\tau}{\mathrm{T}_1}} \, \mathrm{d}\tau \frac{1}{\mathrm{T}_1} = 1 - \alpha$$

respectively of the time-discrete state-space model

$$Z(k+1) = \alpha Z(k) + (1-\alpha)X(k), \quad Z(0) = Z_0$$
$$Y(k) = \mathrm{k_s} Z(k).$$

Fig. 1 depicts the trajectory of the input quantity $X(k)$ of the low-pass filter with $\mathrm{T} = 1$ s. Let $\mathrm{u}_{\bar{x}} = 1$ firstly be the uncertainty attributed to the input values. For illustrational reasons, a smaller input uncertainty $\mathrm{u}_{\bar{x}} = 0.3$ is assumed starting at time $k = 40$.

The low-pass filtering of the input values with the parameters $\mathrm{T}_1 = 10$ s and $\mathrm{k_s} = 1$ yields the result depicted in Fig. 2. Intuitively one might expect that the uncertainty of the filter's state increases monotonously as the low-pass filter is fed with more and more uncertain input values. However, the already existing state and output uncertainties decrease from the time $k = 40$. This is due to the fact that the PT_1 system is stable for every parameterisation. The effect of the first input values with high uncertainty is by and by dominated by the effect of the subsequent input values with less uncertainty.

Since the uncertainty of input values is constant for large k, an equilibrium between the additional uncertainty caused by the coming input values and the degrading state uncertainty emerges. The equilibrium state uncertainty for $k \gg 40$

$$\mathrm{vec}\left(\widehat{\mathbf{U}}_{\bar{\boldsymbol{z}}}\right) = \hat{u}_{\bar{z}}^2 = (1-\alpha^2)^{-1} \cdot (1-\alpha)^2 \cdot 0.3^2$$

$$\Rightarrow \hat{u}_{\bar{y}} = \hat{u}_{\bar{z}} = \sqrt{\hat{u}_{\bar{z}}^2} = 0.067$$

calculated analytically according (9) coincides with the simulation result, which can be read off in Fig. 2.

V. Conclusion

This paper enhances the ISO GUM method to the class of time-discrete dynamic measurement systems. The use of state-space models allows to do this consistently and with only minor changes compared to the original GUM method. It is shown that the evolution of the dynamic system's output uncertainties can be described analytically by a state-space model again. This system is asymptotically stable, if the system describing the the output values is asymptotically stable as well. It is highlighted that the evolution of the state uncertainties and their correlations can exceed all boundaries, reach an equilibrium, or even decrease. The mathematically elegant methods of linear algebra and dynamic systems theory used in this contribution permit an efficient uncertainty evaluation of dynamic systems for real-time applications in line with the standardised GUM method. In the future this approach has to be extended to the handling of model uncertainties.

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