

## ON THE RELATIONS BETWEEN ERROR AND UNCERTAINTY IN MEASUREMENT

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**Abstract:** Are “the error approach” and “the uncertainty approach” really alternative with one another, as the *International Vocabulary of Metrology – Basic and general concepts and associated terms* (VIM3) states, so that assuming the latter implies definitely renouncing to maintain a role for truth in measurement? This paper proposes instead that in the encompassing framework of measurement uncertainty some components of error, and therefore truth, thought of in operative sense, remain and should be taken into account as such.

**Keywords:** measurement uncertainty, measurement error, measuring instrument.

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### 1. INTRODUCTION

The relation between error and uncertainty in measurement is a complex issue, in which philosophical assumptions and modeling techniques are variously intertwined. In a hypothetical spectrum of conceptual options, one extreme is the traditional standpoint according to which the world “is written in mathematical characters” (Galileo), because “numbers are in the world” (Kepler), so that measurement is a process aimed at discovering entities – quantity values – which have an independent existence and therefore operate as references: the better they are approximated, the less the error in the knowledge obtained by measurement. On the opposite side, it is considered that “pure data does not exist” [1], because “data are always theory laden” [2], and in fact “we introduce the numerical concept of weight by setting up a procedure for measuring it. It is *we* who assign numbers to nature. The phenomena themselves exhibit only qualities we observe. Everything numerical [...] is brought in by ourselves when we devise procedures for measurement” [3]. In this view, the lack of pre-existing references leads to characterize measurement results in terms of the belief a subject agrees to attribute to them, expressed as her (un)certainty on such results.

An indicator of this opposition is the usage of the concept of *true quantity value*: once deemed to be an unavoidable, founding element, today it is used – if even it

is used – with much cautiousness and however always stressing that it is an ideal, operatively unknowable concept.

The *International Vocabulary of Metrology – Basic and general concepts and associated terms* (VIM3) [4] further emphasizes the opposition error vs. uncertainty, and in its Introduction presents it in terms of an “Error Approach (sometimes called Traditional Approach or True Value Approach)” and an “Uncertainty Approach”. Such “approaches” are contrasted with each other with respect to their objectives, which “in the Error Approach [is] to determine an estimate of the true value that is as close as possible to that single true value [, whereas] in the Uncertainty Approach [is to assign] an interval of reasonable values to the measurand”. Hence, the opposition error vs. uncertainty is interpreted here as (also) related to the alternative: single values vs. intervals of them, and the concept of true value is pushed to the limit of admitting that while “in the Error Approach [...] a true quantity value is considered unique and, in practice, unknowable [in] the Uncertainty Approach [...] there is not a single true quantity value but rather a set of true quantity values” (in these terms the “Uncertainty Approach” is rather delicate, since the contemporary presence of different truths on the same subjects might appear a violation of the principle of non-contradiction). The change in the article – “the” true value or “a” true value? – might be taken as a lexical symbol of this opposition.

All these elements support the hypothesis that error and uncertainty are incompatible concepts, which cannot be reconciled even at the operative level, so that assuming the latter implies definitely renouncing to maintain a role for error, and therefore truth, in measurement. My claim here is that this incompatibility of “approaches” is inconsistent and unjustified, and that it can and should be overcome, by focusing on the role of measuring instruments and the concept of instrumental measurement uncertainty.

### 2. MEASURING INSTRUMENTS AS CALIBRATED TRANSDUCERS

With the aim of exploring the concepts of measurement error and uncertainty, let us introduce an admittedly simplified model of the empirical core of a measurement process: a *measuring instrument*, i.e., a “device used for making measurements, alone or in conjunction with one or more supplementary devices”, which “may be an indicating measuring instrument or a material measure” [4, 3.1]. In its

turn, an indicating measuring instrument is a “measuring instrument providing an output signal carrying information about the value of the quantity being measured” [4, 3.3] (material measures are not relevant for what follows). The operation described for an indicating measuring instrument is in fact the one performed by a generic transducer, which dynamically produces an output quantity as the effect of its interaction with (a system characterized in particular by) an input quantity.

What does it make a transducer a measuring instrument?

The primary condition is explicitly stated in the definition: the output quantity has to reliably provide information on the quantity being measured. This is a condition of predictability of the input-output behavior, as typically guaranteed by an underlying theory of such behavior and the proper construction and operating procedure of the transducer.

On the other hand, this condition *is not sufficient*. The transducer is a physical device (I am taking into account the case of physical quantities here; what follows may be generalized), whose operation can be modeled as a mapping of quantities,  $q_{out} = \tau_q(q_{in})$ .

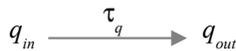


Figure 1 – Simple model of the operation of a transducer.

This cannot be enough, as the purpose of measurement is (in particular) to associate a quantity value to a quantity, the measurand, and the mapping  $\tau_q$  involves only quantities, not quantity values. *How do quantity values appear in the transduction process?*

With the aim of answering this general, and clearly fundamental, question, let us introduce a little bit more complete model of measurement.

Step 1. The measurand  $q_{in}$  (that the transduced quantity is clearly not the general case; I am assuming it just for simplicity) is transduced to an indication  $q_{out}$  (unfortunately, there is a good deal of ambiguity in [4] as to the nature of indications, whether they are quantities or quantity values [5]; I will use here “indication” to denote a quantity and “indication value” to denote a quantity value).

Step 2. The indication  $q_{out}$  is mapped to an indication value  $v_{out}$ ,  $v_{out} = d_{out}(q_{out})$ .

Step 3. A mapping from  $q_{in}$  values to  $q_{out}$  values is available, being the informational counterpart of  $\tau_q$ , which models the empirical operation of the transducer, so that it can be denoted as  $\tau_v$ ,  $v_{out} = \tau_v(v_{in})$ .  $\tau_v$  is provided by the transducer calibration, and is required to be invertible in the measuring interval, corresponding to a monotonic behavior of the transducer in the same interval.

Hence, the measured quantity value  $v_{in}$  for the measurand  $q_{in}$  is obtained as  $v_{in} = m(v_{in}) = \tau_v^{-1}(d_{out}(\tau_q(q_{in})))$ , i.e., by the sequence: transduction to an indication + mapping to an indication value + mapping to a measurand value.

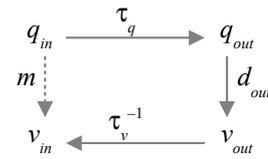


Figure 2 – Measurement as calibrated transduction.

This highlights two further conditions to the applicability of a transducer as a measuring instrument:

$\alpha$ . a mapping from indications to indication values (Step 2) must be available;

$\beta$ . a mapping from measurand values to indication values (Step 3) must be available, i.e., the transducer has to be calibrated.

Both these conditions are specifically significant for our subject. The next two Sections are devoted to discussing them.

### 3. INDICATIONS AND INDICATIONS VALUES

Measurement operates at two layers, by bridging the empirical and the information realms. In the diagram of Fig. 2 such bridge is depicted by the vertical arrows, thus highlighting that empirical operations are not enough to generate a mapping from a quantity to a quantity value. Hence, to the problem – how do quantity values appear in the transduction process? – iterating transductions is not the solution: the transducer output is a quantity, which might be indeed transduced to a third quantity by a second transducer, but this would not make the solution any closer.

In particular, the consideration of the mapping  $v_{out} = d_{out}(q_{out})$ , from indications to indication values, throws some light on the very nature of transducers as core components of measuring systems. *Measurand transduction is designed and performed so to make a reliable mapping quantity  $\rightarrow$  quantity value available.*

In traditional instrumentation, where indications are quantities aimed at being directly perceivable by human beings, the mapping  $d_{out}$  models the operation of such direct perception, where, e.g., the angular position of a needle (the indication) is mapped by the subject who is measuring to an angle value (the indication value). Let us consider the simple situation in which only a small number of indications is allowed. In this case  $d_{out}$  might trivially operate as a labeling: first indication (i.e., first allowed angular position)  $\rightarrow$  “angle1”; second indication  $\rightarrow$  “angle2”; and so on (the condition of invertibility on the calibration mapping  $\tau_v$  would even allow to exploit an unordered set of labels: the only requirement is that  $d_{out}$  is 1-1). Things become more complex because transducers have sometimes a much higher discrimination capability, i.e., they have a high resolution [4, 4.14], and it is usually reasonable that this resolution is not worsened in the reading operation, so that the resolution of the displaying device [4, 4.15] is correspondingly maintained high. Since the resolution of the displaying device might be of the same order of magnitude of, or higher than, the resolution capability of the given human reader, the price to be paid is then that some reading errors might be generated.

The trade-off resolution vs. reliability is clear here.

Most of the current instrumentation, in which the transduction is followed by an ADC stage, even more explicitly highlights the nature of pattern recognition of the mapping  $d_{out}$ , where the set of recognizable patterns, the distinguishable indications, is conventionally mapped to a set of labels, as obtained by the (usually binary) coding of the quantized levels.

The underlying logic remains the same: the mapping from indications to indication values results from a design decision, which can be thoroughly planned. The conclusion is that in principle each indication can have associated its own indication value, which is then true-by-design, or “operatively true”. In reference to the above mentioned Kuhn’s consideration [1], the mapping  $d_{out}$  is designed so to be able to generate (operatively) *pure data*.

#### 4. MEASURAND VALUES AND INDICATIONS VALUES

For calibrating the transducer, i.e., performing Step 3 as described above, the quantity value  $v_{in}$  must be known for some input quantities  $q_{in}$ . In the case of calibration, the diagram of Fig. 2 becomes as in Fig. 3, where this condition refers to the mapping  $d_r$ , as applied to specific quantities  $r_i$ .

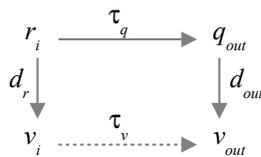


Figure 3 – Transducer calibration.

As in the case of  $d_{out}$ , the mapping  $d_r$  can be designed, particularly because it can be applied to controllable reference quantities  $r_i$ , as realized by appropriate measurement standards. Indeed, performing  $d_r$  is not part of a measurement, but of the so called *scale construction*, aimed at exploiting the available knowledge on the general quantity under measurement by (i) establishing the scale type (i.e., nominal, ordinal, ...), (ii) identifying an appropriate set of reference quantities  $r_i$  accordingly (for more-than-ordinal scales a single measurement unit is enough), and properly associating the quantity values  $v_i$  to them. This is the basic target of the representational theories of measurement [6], which might be then more properly called theories of scale construction [7; 8].

Again as in the case of  $d_{out}$ , a trade-off resolution vs. reliability applies to the mapping  $d_r$ , and therefore the association reference  $\rightarrow$  reference value usually becomes more and more problematic as the metrological traceability chain [4, 2.42] becomes longer, so that the quantity values realized by working measurement standards [4, 5.7] have typically an uncertainty associated. But, exactly as above, in principle each reference can have associated its own reference value, which is then true-by-design, or “operatively true”.

## 5. SOME CONCLUSIONS

The concept of instrumental measurement uncertainty – the “component of measurement uncertainty arising from a measuring instrument or measuring system in use” [4, 4.24] – is clearly a fundamental one for its both theoretical and operative implications on measurement. It is a complex “component of measurement uncertainty”, in its turn including several components, specifically related to measurement precision [4, 2.15], measurement trueness [4, 2.14], and measurement accuracy [4, 2.13].

The previous discussion leads to consider that the instrumental measurement uncertainty includes some components of error, referred to the (operative) true values of references exploited in calibration and indications obtained in transduction. At least for such quantities, a general trade-off is appropriately taken into account between *certainty of truth* and *specificity of truth*. This mirrors the probabilistic relation between confidence intervals and confidence levels: for a given probability distribution, which models the available information, as the interval width increases the level also increases, i.e., the conveyed information becomes less specific but more certain. The same applies to the construction of reference scales and indication scales.

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