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TWO CALIBRATION MODELS

G. Wimmer[1,2,3] *and V. Witkovský*[4]

[1]Institute of Mathematics and Statistics, Masaryk University Brno, Czech Republic
[2]Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovak Republic
[3]Faculty of Natural Sciences, Matej Bel University, Banská Bystrica, Slovak Republic
wimmer@mat.savba.sk
[4]Institute of Measurement Science, Slovak Academy of Sciences, Bratislava, Slovak Republic,
witkovsky@savba.sk

Abstract: Analyzed is the comparative calibration problem in the case when linear relationship is assumed between two considered measuring devices. The first method for obtaining the approximate confidence region for unknown parameters of the calibration line applies the maximum likelihood estimators of the unknown parameters. The second method is based on estimation of the calibration line via replicated errors-in-variables model. Essential point in this approach is the use of the F-approximation of the distribution of the F-statistic suggested by Kenward and Roger (1997). The contribution shows also the interval estimators for the multiple-use calibration using both methods and enables to compare them.

Keywords: Calibration problem, multiple-use calibration, maximum likelihood estimator, Kenward-Roger approximation.

1. MAXIMUM LIKELIHOOD ESTIMATORS OF THE CALIBRATION LINE PARAMETERS

Let us have measurements $X_{11}, Y_{11}, ..., X_{n1}, Y_{n1}$. We suppose that the measurements are normally distributed, independent and it is valid that the mean value of Y_{i1} is

$$\mathcal{E}(Y_{i1}) = a + b\mu_i, \quad i = 1, 2, ..., n,$$

where μ_i is the mean value of X_{i1} and a, b are the unknown parameters of the calibration line (more see in [2]). This group of $2n$ mesurements we consider as one experiment. The likelihood function of the random vector $(X_{11}, Y_{11}, ..., X_{n1}, Y_{n1})'$ is

$$L(x_{11}, y_{11}, ..., x_{n1}, y_{n1}; a, b, \sigma_x^2, \sigma_y^2, \mu_1, ..., \mu_n) = \prod_{i=1}^n f_i(x_{i1}; \mu_i, \sigma_x^2) \prod_{j=1}^n g_j(y_{j1}; a + b\mu_j, \sigma_y^2) =$$

$$= \frac{1}{(2\pi)^n \sigma_x^n \sigma_y^n} e^{-\frac{1}{2\sigma_x^2}\sum\limits_{i=1}^n (x_{i1}-\mu_i)^2 - \frac{1}{2\sigma_y^2}\sum\limits_{j=1}^n (y_{j1}-a-b\mu_j)^2}.$$

This experiment we repeat independently $m-$times. The $r-$ th experiment is modelled by the random vector $(X_{1r}, Y_{1r}, ..., X_{nr}, Y_{nr})'$. The likelihood function of the whole calibration experiment (consisting of m experiments) is

$$L(x_{11}, y_{11}, ..., x_{nm}, y_{nm}; a, b, \sigma_x^2, \sigma_y^2, \mu_1, ..., \mu_n) = \frac{1}{(2\pi)^{nm} \sigma_x^{nm} \sigma_y^{nm}} e^{-\frac{1}{2\sigma_x^2}\sum\limits_{i=1}^n \sum\limits_{j=1}^m (x_{ij}-\mu_i)^2 - \frac{1}{2\sigma_y^2}\sum\limits_{i=1}^n \sum\limits_{j=1}^m (y_{ij}-a-b\mu_i)^2}.$$

The maximum likelihood estimators $\hat{a}(X_{11}, ..., Y_{nm})$, $\hat{b}(X_{11}, ..., Y_{nm})$, $\hat{\sigma}_x^2(X_{11}, ..., Y_{nm})$, $\hat{\sigma}_y^2(X_{11}, ..., Y_{nm})$, $\hat{\mu}_1(X_{11}, ..., Y_{nm})$,..., $\hat{\mu}_n(X_{11}, ..., Y_{nm})$ are solutions of the likelihood equations

$$\sum_{i=1}^n \sum_{j=1}^m (Y_{ij} - \hat{a} - \hat{b}\hat{\mu}_i) = 0, \quad \sum_{i=1}^n \sum_{j=1}^m (Y_{ij} - \hat{a} - \hat{b}\hat{\mu}_i)\hat{\mu}_i = 0,$$

$$\sum_{i=1}^n \sum_{j=1}^m (X_{ij} - \hat{\mu}_i)^2 = mn\hat{\sigma}_x^2, \sum_{i=1}^n \sum_{j=1}^m (Y_{ij} - \hat{a} - \hat{b}\hat{\mu}_i)^2 = mn\hat{\sigma}_y^2,$$

$$\hat{\sigma}_y^2 \sum_{j=1}^m (X_{1j} - \hat{\mu}_1) = -\hat{\sigma}_x^2 \hat{b} \sum_{j=1}^m (Y_{1j} - \hat{a} - \hat{b}\hat{\mu}_1)$$

$$\vdots$$

$$\hat{\sigma}_y^2 \sum_{j=1}^m (X_{nj} - \hat{\mu}_n) = -\hat{\sigma}_x^2 \hat{b} \sum_{j=1}^m (Y_{nj} - \hat{a} - \hat{b}\hat{\mu}_n).$$

It is valid

$$\sqrt{m}\begin{pmatrix} \hat{a} - a \\ \hat{b} - b \end{pmatrix} \xrightarrow{\mathcal{D}} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{\Sigma}\right),$$

(convergence in distribution). The asymptotic covariance matrix of the likelihood estimator $(\hat{a}, \hat{b})'$ is

$$\mathbf{\Sigma} = \frac{b^2\sigma_x^2 + \sigma_y^2}{m\left(n\sum_{i=1}^n \mu_i^2 - \left(\sum_{j=1}^n \mu_j\right)^2\right)} \times \begin{pmatrix} \sum_{i=1}^n \mu_i^2 & -\sum_{i=1}^n \mu_i \\ -\sum_{i=1}^n \mu_i & n \end{pmatrix}.$$

We use the approximate $\hat{\mathbf{\Sigma}}$

$$\hat{\mathbf{\Sigma}} = \frac{\hat{b}_{real}^2 \hat{\sigma}_{x,real}^2 + \hat{\sigma}_{y,real}^2}{m\left(n\sum_{i=1}^n \hat{\mu}_{i,real}^2 - \left(\sum_{j=1}^n \hat{\mu}_{j,real}\right)^2\right)} \times$$

$$\begin{pmatrix} \sum_{i=1}^n \hat{\mu}_{i,real}^2 & -\sum_{i=1}^n \hat{\mu}_{i,real} \\ -\sum_{i=1}^n \hat{\mu}_{i,real} & n \end{pmatrix} \quad (1)$$

($\hat{b}_{real}$ is the realization of the likelihood estimator $\hat{b}$), and we obtain the maximum likelihood (approximate, asymptotic) $(1-\alpha)-$confidence region for $(a,b)'$

$$\mathcal{C}_{(1-\alpha)}^{(ML)} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : \begin{pmatrix} \hat{a}-a \\ \hat{b}-b \end{pmatrix}' \hat{\mathbf{\Sigma}}^{-1} \begin{pmatrix} \hat{a}-a \\ \hat{b}-b \end{pmatrix} \leqq \chi_2^2(1-\alpha) \right\}$$

($\chi_2^2(1-\alpha)$ is the $(1-\alpha)-$quantile of the χ^2 distribution with 2 degrees of freedom).

2. ESTIMATION OF THE CALIBRATION LINE PARAMETERS VIA REPLICATED ERRORS-IN-VARIABLES MODEL

Calibration experiment we can model using EIV model (errors in variables model)

$$Y_i = \alpha + \beta\mu_i + \varepsilon_i, \quad X_i = \mu_i + \delta_i$$

($\varepsilon_i \sim N(0,\sigma_Y^2),\ \delta_i \sim N(0,\sigma_X^2)$, independent). Written in another way

$$\mathcal{E}(Y_i) = \alpha + \beta\mu_i, \quad \mathcal{E}(X_i) = \mu_i.$$

Vectors of errorless measurements realized using instruments A a B are $\boldsymbol{\mu} = (\mu_1,\mu_2,\dots,\mu_n)'$ and $\boldsymbol{\nu} = (\nu_1,\nu_2,\dots,\nu_n)'$. Vector of measurements with instrument A is $\mathbf{X}_{n,1} \sim N(\boldsymbol{\mu};\sigma_x^2\boldsymbol{I}_{n,n})$. Vector of measurements with instrument B is $\mathbf{Y}_{n,1} \sim N(\boldsymbol{\nu};\sigma_y^2\boldsymbol{I}_{n,n})$. We obtain the model

$$\begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \end{pmatrix} \sim N\left[\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix}, \begin{pmatrix} \sigma_x^2\boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \sigma_y^2\boldsymbol{I} \end{pmatrix}\right]$$

with condition on parameters

$$\boldsymbol{\nu} = a\mathbf{1}_{n,1} + b\boldsymbol{\mu},$$

where $\mathbf{1}_{n,1} = (1,1,\dots,1)'$. First we linearize the model using Taylor series in a neighborhood of $\boldsymbol{\mu}_0 = (\mu_{01},\mu_{02},\dots,\mu_{0n})'$ a b_0 (some values near the reality $\boldsymbol{\mu}$ a b). Now $\boldsymbol{\mu} = \boldsymbol{\mu}_0 + \delta\boldsymbol{\mu},\ b = b_0 + \delta b$ and the new model parameters are $\delta\boldsymbol{\mu} = (\delta\mu_1,\delta\mu_2,\dots,\delta\mu_n)',\ \boldsymbol{\nu},\ a,\ \delta b, \sigma_x^2,\ \sigma_y^2$. We get the (approximative) linear regression model

$$\begin{pmatrix} \boldsymbol{X}-\boldsymbol{\mu}_0 \\ \boldsymbol{Y} \end{pmatrix} \sim N\left[\begin{pmatrix} \delta\boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix}, \begin{pmatrix} \sigma_x^2\boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \sigma_y^2\boldsymbol{I} \end{pmatrix}\right] \quad (2)$$

with (linear) conditions on parameters

$$b_0\boldsymbol{\mu}_0 + (b_0\boldsymbol{I} \vdots -\boldsymbol{I})\begin{pmatrix} \delta\boldsymbol{\mu} \\ \boldsymbol{\nu} \end{pmatrix} + (\mathbf{1},\boldsymbol{\mu}_0)\begin{pmatrix} a \\ \delta b \end{pmatrix} = \mathbf{0}. \quad (3)$$

Dispersions σ_x^2 and σ_y^2 are unknown. One possibility to estimate them are the $(\sigma_{x0}^2,\sigma_{y0}^2)$-MINQUE estimators (Minimum Norm Quadratic Unbiased Estimator). As this estimators do not exist in model (2) - (3), we need to repeat the whole experiment m times independently.

The repeated measurements are $\boldsymbol{X}_j = (X_{j1},\dots,X_{jn})'$, $\boldsymbol{Y}_j = (Y_{j1},\dots,Y_{jn})',\ \ j=1,\dots,m$. The best linear unbiased estimators of $\boldsymbol{\mu},\boldsymbol{\nu},\ a,\delta b$ in replicated model are (see [3])

$$\hat{\boldsymbol{\mu}} = \bar{\boldsymbol{X}} + \frac{b_0\sigma_x^2}{b_0^2\sigma_x^2+\sigma_y^2}\boldsymbol{M}_{[1,\mu_0]}(\bar{\boldsymbol{Y}} - b_0\bar{\boldsymbol{X}}), \quad (4)$$

$$\hat{\boldsymbol{\nu}} = \bar{\boldsymbol{Y}} - \frac{\sigma_y^2}{b_0^2\sigma_x^2+\sigma_y^2}\boldsymbol{M}_{[1,\mu_0]}(\bar{\boldsymbol{Y}} - b_0\bar{\boldsymbol{X}}), \quad (5)$$

$$\begin{pmatrix} \hat{a} \\ \delta\hat{b} \end{pmatrix} = \begin{pmatrix} n & \mathbf{1}'\boldsymbol{\mu}_0 \\ \boldsymbol{\mu}_0'\mathbf{1} & \boldsymbol{\mu}_0'\boldsymbol{\mu}_0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1}'(\bar{\boldsymbol{Y}} - b_0\bar{\boldsymbol{X}}) \\ \boldsymbol{\mu}_0'(\bar{\boldsymbol{Y}} - b_0\bar{\boldsymbol{X}}) \end{pmatrix}, \quad (6)$$

with the covariance matrix

$$\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \frac{b_0^2\sigma_x^2+\sigma_y^2}{m}\begin{pmatrix} n & \mathbf{1}'\boldsymbol{\mu}_0 \\ \boldsymbol{\mu}_0'\mathbf{1} & \boldsymbol{\mu}_0'\boldsymbol{\mu}_0 \end{pmatrix}^{-1},$$

where

$$\bar{\boldsymbol{X}} = \frac{1}{m}\sum_{j=1}^m \boldsymbol{X}_j, \quad \bar{\boldsymbol{Y}} = \frac{1}{m}\sum_{j=1}^m \boldsymbol{Y}_j$$

and

$$\boldsymbol{M}_{[1,\mu_0]} = \boldsymbol{I} - [\mathbf{1},\boldsymbol{\mu}_0]\left([\mathbf{1},\boldsymbol{\mu}_0]'[\mathbf{1},\boldsymbol{\mu}_0]\right)^{-1}[\mathbf{1},\boldsymbol{\mu}_0]'.$$

$(\sigma_{x0}^2,\sigma_{y0}^2)$-MINQUE estimators of σ_x^2 a σ_y^2 in replicated model are

$$\begin{pmatrix} \hat{\sigma}_x^2 \\ \hat{\sigma}_y^2 \end{pmatrix} = \frac{1}{n(m-1)}\left[\boldsymbol{I}_{2,2} - c_0\begin{pmatrix} b_0^4\sigma_{x0}^4 & b_0^2\sigma_{x0}^4 \\ b_0^2\sigma_{y0}^4 & \sigma_{y0}^4 \end{pmatrix}\right]\begin{pmatrix} \hat{\kappa}_1 \\ \hat{\kappa}_2 \end{pmatrix}, \quad (7)$$

where

$$c_0 = \frac{n-2}{(b_0^4\sigma_{x0}^4+\sigma_{y0}^4)(mn-2) + 2b_0^2\sigma_{x0}^2\sigma_{y0}^2(m-1)n},$$

$$\hat{\kappa}_1 = \sum_{j=1}^m (\boldsymbol{X}_j - \bar{\boldsymbol{X}})'(\boldsymbol{X}_j - \bar{\boldsymbol{X}}) + m(\bar{\boldsymbol{X}} - \hat{\boldsymbol{\mu}})'(\bar{\boldsymbol{X}} - \hat{\boldsymbol{\mu}}),$$

$$\hat{\kappa}_2 = \sum_{j=1}^m (\boldsymbol{Y}_j - \bar{\boldsymbol{Y}})'(\boldsymbol{Y}_j - \bar{\boldsymbol{Y}}) + m(\bar{\boldsymbol{Y}} - \hat{\boldsymbol{\nu}})'(\bar{\boldsymbol{Y}} - \hat{\boldsymbol{\nu}}).$$

The covariance matrix of the estimators (7) (local in the values $(\sigma_{x0}^2,\sigma_{y0}^2)$) is

$$\boldsymbol{W} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} =$$

$$\frac{2}{n(m-1)}\left[\boldsymbol{I}_{2,2} - c_0\begin{pmatrix} b_0^4\sigma_{x0}^4 & b_0^2\sigma_{x0}^4 \\ b_0^2\sigma_{y0}^4 & \sigma_{y0}^4 \end{pmatrix}\right]\begin{pmatrix} \sigma_{x0}^4 & 0 \\ 0 & \sigma_{y0}^4 \end{pmatrix}. \quad (8)$$

A natural choice of the intial values resulting from the measurements can be as follows

$$\hat{\boldsymbol{\mu}}_0 = \bar{\boldsymbol{X}}, \quad \hat{b}_0 = \frac{n\bar{\boldsymbol{X}}'\bar{\boldsymbol{Y}} - (\mathbf{1}'\bar{\boldsymbol{X}})(\mathbf{1}'\bar{\boldsymbol{Y}})}{n\bar{\boldsymbol{X}}'\bar{\boldsymbol{X}} - (\mathbf{1}'\bar{\boldsymbol{X}})^2},$$

$$\hat{\sigma}_{x0}^2 = \frac{1}{n(m-1)}\sum_{i=1}^n\sum_{j=1}^m (X_{ji} - \bar{X}_i)^2,$$

$$\hat{\sigma}_{y0}^2 = \frac{1}{n(m-1)}\sum_{i=1}^n\sum_{j=1}^m (Y_{ji} - \bar{Y}_i)^2. \quad (9)$$

Further is computed $\hat{a},\hat{b}$, from (6), $\hat{\boldsymbol{\mu}}$ from (4), $\hat{\boldsymbol{\nu}}$ from (5), $\hat{\sigma}_x^2$ and $\hat{\sigma}_y^2$ from (7). The estimation procedure is iterative till the convergence is reached (usually in 4-5 steps). After the procedure is finished, computed is the covariance matrix $\boldsymbol{W}$ according to (8).

We have obtained

$$\begin{pmatrix} \hat{a}-a \\ \hat{b}-b \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{b_0^2\sigma_x^2+\sigma_y^2}{m}\begin{pmatrix} n & \mathbf{1}'\boldsymbol{\mu}_0 \\ \boldsymbol{\mu}_0'\mathbf{1} & \boldsymbol{\mu}_0'\boldsymbol{\mu}_0 \end{pmatrix}^{-1}\right).$$

To emphasize its dependence on the parameters (σ_x^2,σ_y^2), we will alternatively denote the covariance matrix of the distribution also by $\boldsymbol{\Phi}(\sigma_x^2,\sigma_y^2)$. If the parameters σ_x^2,σ_y^2 were known, $(1-\alpha)-$confidence region for parameters a,b is

$$\mathcal{C}^{*}_{(1-\alpha)}=\left\{\begin{pmatrix}a\\b\end{pmatrix}:\frac{m}{b_0^2\sigma_x^2+\sigma_y^2}Q^{(EIV)}\leqq\chi_2^2(1-\alpha)\right\}.$$

where

$$Q^{(EIV)}=\begin{pmatrix}\hat{a}-a\\\hat{b}-b\end{pmatrix}'\begin{pmatrix}n & \mathbf{1}'\boldsymbol{\mu}_0\\\boldsymbol{\mu}_0'\mathbf{1} & \boldsymbol{\mu}_0'\boldsymbol{\mu}_0\end{pmatrix}\begin{pmatrix}\hat{a}-a\\\hat{b}-b\end{pmatrix}.$$

That is why σ_x^2,σ_y^2 are unknown, we apply the procedure suggested by Kenward and Roger [1] to obtain a Wald-type statistics and its approximation by F distribution. This procedure was suggested for small range of measured data (in our case little m,n). Kenward and Roger proposed a modified estimator of the matrix $\boldsymbol{\Phi}$ of the form

$$\hat{\boldsymbol{\Phi}}_A=\hat{\boldsymbol{\Phi}}-\sum_{i=1}^{2}\sum_{j=1}^{2}w_{ij}\frac{\partial^2\boldsymbol{\Phi}}{\partial\sigma_i^2\partial\sigma_j^2}$$

where $\hat{\boldsymbol{\Phi}}=\boldsymbol{\Phi}(\hat{\sigma}_x^2,\hat{\sigma}_y^2), w_{ij}=\{\mathbf{W}\}_{ij}$ ($\mathbf{W}$ is given in (8)) and $\sigma_1^2=\sigma_x^2,\sigma_2^2=\sigma_y^2$. After computations we obtain

$$\sum_{i=1}^{2}\sum_{j=1}^{2}w_{ij}\frac{\partial^2\boldsymbol{\Phi}}{\partial\sigma_i^2\partial\sigma_j^2}=0,$$

so $\hat{\boldsymbol{\Phi}}_A=\hat{\boldsymbol{\Phi}}$. The moodified estimator $\hat{\boldsymbol{\Phi}}_A$ is recommended to use in the statistics

$$F=\frac{1}{2}\begin{pmatrix}\hat{a}-a\\\hat{b}-b\end{pmatrix}'\hat{\boldsymbol{\Phi}}_A^{-1}\begin{pmatrix}\hat{a}-a\\\hat{b}-b\end{pmatrix}.$$

Further approximation of F is in such a way that λF is $F_{2,u}$ distributed (Fisher-Snedecor distribution with 2 and u degrees of freedom). Analogically as Kenward and Roger considerations after tedious computations we obtain

$$\lambda=1,$$

and

$$u=(mn-2)+\frac{2b_0^2\hat{\sigma}_x^2\hat{\sigma}_y^2(m-1)n}{b_0^4\hat{\sigma}_x^4+\hat{\sigma}_y^4}.$$

So,

$$\mathcal{C}^{(EIV)}_{(1-\alpha)}=\left\{\begin{pmatrix}a\\b\end{pmatrix}:\frac{m}{2(b_0^2\hat{\sigma}_x^2+\hat{\sigma}_y^2)}Q^{(EIV)}\leqq F_{2,u}(1-\alpha)\right\},$$

is the $(1-\alpha)-$confidence region for $(a,b)'$, ($F_{t,u}(1-\alpha)$ is the $(1-\alpha)-$quantile of $F_{t,u}$ distribution).

2.1. SCHEFFÉ-TYPE CONFIDENCE REGION FOR THE CALIBRATION LINE

If the true values of calibration line coefficients are a and b, then the following (approximative) distribution is valid:

$$F=\frac{1}{2}\frac{m}{b_0^2\hat{\sigma}_x^2+\hat{\sigma}_y^2}Q^{(EIV)}\sim F_{2,u}.$$

From that we get

$$\Pr\left\{\frac{1}{2}\frac{m}{b_0^2\hat{\sigma}_x^2+\hat{\sigma}_y^2}Q^{(EIV)}\leq F_{2,u}(1-\alpha)\right\}=1-\alpha.$$

By applying the Scheffé's Theorem we directly get the $100\times(1-\alpha)\%$-confidence region for the calibration line $a+b\mu$ for all μ:

$$\Pr\left\{\left|(\hat{a}+\hat{b}\mu)-(a+b\mu)\right|\leq\Delta(\mu)\right\}=1-\alpha$$

with

$$\Delta(\mu)=\sqrt{2F_{2,u}(1-\alpha)\frac{b_0^2\hat{\sigma}_x^2+\hat{\sigma}_y^2}{m}\left(\frac{1}{n}+\frac{(\mu-\bar{\mu}_0)^2}{\mu_0'\mu_0-n\bar{\mu}_0^2}\right)},$$

where $\bar{\mu}_0=(1'\mu_0)/n$. This could be directly used for the *multiple-use linear univariate calibration*, i.e. for measuring with calibrated device.

2.2. MULTIPLE-USE CALIBRATION - MEASURING WITH CALIBRATED DEVICE

We will assume that the future measurement realized by the calibrated (less precise) measurement device A, say x, is a realization of a random variable X, distributed as $X\sim N(\mu_x,\sigma_x^2)$, where μ_x represents unobservable true value of the measurand.

First, we suggest to construct the approximate $(1-\gamma)$-confidence region for the calibration line, for small significance level $\gamma\in(0,1)$, chosen by the user, according to (9).

Second, for small significance level $\alpha\in(0,1)$, we suggest to construct the approximate $(1-\alpha)$-confidence interval for μ_x. For that we suggest to construct $t-$ statistic with approximate t_v Student's t distribution.

$$t=\frac{X-\mu_x}{\hat{\sigma}_x}\overset{approx}{\sim}t_v,$$

where $v=2\hat{\sigma}_x^4/w_{11}$ are the approximated degrees of freedom. This leads to the approximate $(1-\alpha)-$confidence interval for unobservable value μ_x:

$\mu_x\in\langle x-\hat{\sigma}_x t_v(1-\alpha/2),x+\hat{\sigma}_x t_v(1-\alpha/2)\rangle=\langle\mu_{xl},\mu_{xu}\rangle$ ($t_v(1-\alpha/2)$ is the $(1-\alpha/2)-$ quantile of Student's t distribution with v degrees of freedom). The suggested interval estimator for ν_x is $\langle\nu_{xl},\nu_{xu}\rangle$ and is given as the intersection of the bounds of the $(1-\gamma)-$confidence region for the whole calibration line $a+b\mu$ and the limits of the $(1-\alpha)-$confidence interval $\langle\mu_{xl},\mu_{xu}\rangle$ for μ_x. In fact,

$$\nu_{xl}=\hat{a}+\hat{b}\mu_{xl}-\Delta(\mu_{xl}),$$
$$\nu_{xu}=\hat{a}+\hat{b}\mu_{xu}+\Delta(\mu_{xu}). \qquad (10)$$

Using Bonferroni's inequality, $\langle\nu_{xl},\nu_{xu}\rangle$ is an (approximative) at least $1-(\alpha+\gamma)-$ confidence interval for the (unobservable) value ν_x. Simulations indicate that the confidence interval $\langle\nu_{xl},\nu_{xu}\rangle$ is conservative, "safe" and appropriate for metrology.

3. MMULTIPLE-USE CALIBRATION USING MAXIMUM LIKELIHOOD APPROACH

In this section we everywhere use the maximum likelihood estimators. It is valid

$$\chi^2=\begin{pmatrix}\hat{a}-a\\\hat{b}-b\end{pmatrix}'\hat{\boldsymbol{\Sigma}}^{-1}\begin{pmatrix}\hat{a}-a\\\hat{b}-b\end{pmatrix}\sim\chi_2^2$$

($\hat{\boldsymbol{\Sigma}}$ is given in (1)), and from that we get

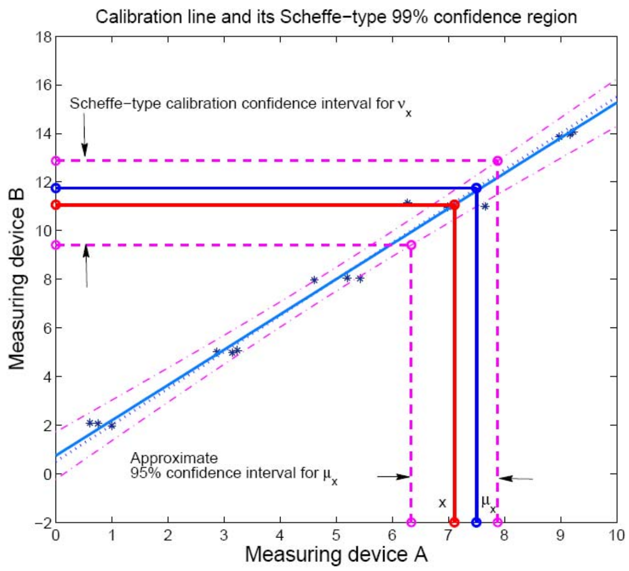

Fig. 1 Calibration line and its Scheffé-type confidence region. The thick dotted line represents the true calibration line, the solid line is the estimated calibration line together with the limits of the 99% confidence region (dashed-dotted lines). The dashed lines represents the Scheffé-type interval estimator for $\nu_x = a + b\mu_x$, where $\mu_x = 7.5$, based on $x = 7.1097$, the realization of random variable $X \sim N(7.5, 0.15)$.

$$\Pr\left\{\begin{pmatrix} \hat{a}-a \\ \hat{b}-b \end{pmatrix}' \hat{\mathbf{\Sigma}}^{-1} \begin{pmatrix} \hat{a}-a \\ \hat{b}-b \end{pmatrix} \leqq \chi_2^2(1-\alpha)\right\} = 1-\alpha.$$

The $100 \times (1-\alpha)\%$-confidence region for the calibration line $a + b\mu$ for all μ is

$$\Pr\left\{\left|(\hat{a}+\hat{b}\mu)-(a+b\mu)\right| \leq \Theta(\mu)\right\} = 1-\alpha$$

with

$$\Theta(\mu) = \sqrt{\chi_2^2(1-\alpha)\left(\{\hat{\mathbf{\Sigma}}\}_{11} + 2\mu\{\hat{\mathbf{\Sigma}}\}_{12} + \mu^2\{\hat{\mathbf{\Sigma}}\}_{22}\right)}.$$

The approximate $(1-\alpha)-$ confidence interval for unobservable value μ_x is

$$\mu_x \in \langle x - \hat{\sigma}_x t_v(1-\alpha/2), x + \hat{\sigma}_x t_v(1-\alpha/2)\rangle = \langle \mu_{xl}, \mu_{xu}\rangle$$

where v is now $2\hat{\sigma}_x^4/\{\hat{\mathbf{\Sigma}}\}_{11}$. Finally the suggested interval estimator for ν_x , i.e. the (approximative) at least $1-(\alpha+\gamma)$ -confidence interval for the (unobservable) value ν_x is $\langle \nu_{xl}, \nu_{xu}\rangle$ with

$$\nu_{xl} = \hat{a} + \hat{b}\mu_{xl} - \Theta(\mu_{xl}),$$
$$\nu_{xu} = \hat{a} + \hat{b}\mu_{xu} + \Theta(\mu_{xu}). \qquad (11)$$

The estimators given in (10) and (11) would be compared.

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