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MATRIX INTERPRETATION OF THE UNCERTAINTY PROPAGATION FOR THE ITS-90 REALISATION

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Abstract: Matrix approach to the propagation of uncertainties in the temperature scale ITS-90 realisation based on the Guide to the Expression of Uncertainty in Measurement (GUM) and the Supplement 2 to the GUM. This approach allows to include the correlations between the SPRT resistances from calibration at the defining fixed points in the calculation. The contribution of correlation to the overall uncertainty is presented too.

Keywords: Uncertainty, Correlation, Platinum resistance thermometers (SPRT), ITS-90, Matrix algebra

1. INTRODUCTION

Uncertainty propagation of the temperature scale realisation ITS-90 using standard platinum resistance thermometers (SPRTs) calibrated at the defining fixed points (DFPs) is presented with an overview of different approaches in the BIPM document [1]. One approach provided in this document is based on orthogonal interpolation functions in details. That approach is also introduced in [2, 3, 4, 5]. Other approaches are based on the GUM [6, 7].

Part of approaches presents in accordance with the GUM [6] the uncertainty propagation for SPRTs with summation formulas [8, 9, 10, 11, 12, 13, 14, 15]. Most authors, however, do not consider the correlation between the resistances at the DFPs for calibration. This paper reflects the correlation between the resistances at the DFPs especially in [8, 9, 10, 11]. Another approach is based on the Supplement 2 to the GUM [7]. This approach uses matrix algebra. Matrix approach was presented by authors of this article in [16, 17] but only for the temperature range from 273,15 K to 933,473 K. Rosenkranz [8] followed these works and in accordance with [7] elaborated procedure for uncertainty propagation of the temperature scale in matrix form. Correlations between SPRT resistances at the DFPs were not considered there. Correlations are apparent, for example, using the same value of SPRT resistance at the triple point of water (TPW) in temperature measurement as it was measured during calibration.

In addition, the SPRT resistances at DFPs can be correlated with regard to some of the same conditions of calibration and temperature measurement in the laboratory. Another possible approach is based on the Supplement 1 to the GUM [20].

This paper introduces an approach using matrix algebra according to the Supplement 2 to the GUM [6, 7]. Presented procedure for determining the temperature uncertainty takes into account correlations between SPRT resistances at DFPs from calibration.

2. UNCERTAINTIES OF TEMPERATURE SCALE REALISATION

Temperature T according to the ITS-90 is determined by the inverse reference function

$$T = f\left(W_{\mathrm{r}}\right) \tag{1}$$

For different subranges are functions of (1) given in [21]. Function Wr is given by

$$\begin{aligned} W_{\mathrm{r}} - W + \textstyle\sum_{i=1}^{N} a_i\, f_i(W) = 0 \\ W - \frac{R}{R_{\mathrm{TPW}}} = 0 \end{aligned} \tag{2}$$

where R is the SPRT resistance at the temperature T and R_{TPW} is the SPRT resistance at TPW, f_i(W) are functions of the individual temperature subranges [20], a_i are the coefficients of deviation function obtained by SPRT calibration at DFPs and

$a_i = g_1(R_{\mathrm{TPW},1}, \ldots, R_{\mathrm{TPW},N}, R_{\mathrm{DFP},1}, \ldots, R_{\mathrm{DFP},N})$ resp.
$a_i = g_2(W_{\mathrm{DFP},1}, W_{\mathrm{DFP},2}, \ldots, W_{\mathrm{DFP},N})$.

If equation (2) is applied for N DFPs

$$\begin{pmatrix} \Delta W_{\mathrm{DFP1}} \\ \vdots \\ \Delta W_{\mathrm{DFPN}} \end{pmatrix} = \begin{pmatrix} f_1(W_{\mathrm{DFP1}}) & \cdots & f_N(W_{\mathrm{DFP1}}) \\ \vdots & \ddots & \vdots \\ f_N(W_{\mathrm{DFP1}}) & \cdots & f_N(W_{\mathrm{DFP}N}) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \tag{3}$$

where $\Delta W_{\mathrm{DFP}i} = W_{\mathrm{r,DFP}i} - W_{\mathrm{DFP}i}$

Equations (3) rewritten in matrix form

$$\Delta \boldsymbol{W} = \boldsymbol{M}_{\mathrm{DFP}}\, \boldsymbol{a} \tag{4}$$

Then the coefficients are determined as

$$\boldsymbol{a} = \boldsymbol{M}_{\mathrm{DFP}}^{-1}\, \Delta \boldsymbol{W} \tag{5}$$

because there exists $\boldsymbol{M}_{\mathrm{DFP}}^{-1}$.

3. UNCERTAINTY PROPAGATION FOR ITS-90

The uncertainty of temperature according to the equation (1) is

$$u(T) = \left.\frac{\partial T}{\partial W_r(T)}\right|_{W_r} u(W_r) \tag{6}$$

For determination of uncertainty $u(W_r)$ model (2) is written in matrix form in accordance with the Supplement 2 to the GUM [7]

$$W_r = W + \boldsymbol{a}^{\mathrm{T}} \boldsymbol{f}(W) \tag{7}$$

According to [7], paragraph 6.2.1 is

$$u^2(W_r) = \boldsymbol{C}_x \boldsymbol{U}_{W,\boldsymbol{a}}\, \boldsymbol{C}_x^{\mathrm{T}} \tag{8}$$

where

$$\boldsymbol{C}_x = \left(1 - \sum_{i=1}^{N} a_i \frac{\partial f_i(W)}{\partial W}, f_1(W), \cdots, f_N(W)\right). \tag{9}$$

Further, because $W_r \approx W$, the derivative $\frac{\partial W_r}{\partial W} \approx 1$ and the term $1 - \sum_{i=1}^{N} a_i \frac{\partial f_i(W)}{\partial W} \approx 1$. Specific forms of the matrix of sensitive coefficients $\boldsymbol{C}$ for individual subranges are given in ITS-90 [8]. Then

$$\boldsymbol{C}_x = \left(1, f_1(W) \cdots, f_N(W)\right). \tag{10}$$

As Rosenkranz presents in [19], for determination of covariance matrix $\boldsymbol{U}_{W,\boldsymbol{a}}$ in equation (9) is assumed model

$$\begin{pmatrix} W - \dfrac{R}{R_{\mathrm{TPW}}} \\ \\ \Delta \boldsymbol{W}_{\mathrm{DFP}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{0}_{1\times N} \\ \boldsymbol{M}_{\mathrm{DFP},N\times N}, \end{pmatrix} \boldsymbol{a}_{N\times 1} = \boldsymbol{0}_{(1+N)\times 1} \tag{11}$$

Then for $\boldsymbol{Y} = (W, a_2, \cdots, a_N)^{\mathrm{T}} = (W, \boldsymbol{a}^{\mathrm{T}})^{\mathrm{T}}$ and $\boldsymbol{X} = (R, R_{\mathrm{TPW}}, W_{\mathrm{DFP1}}, W_{\mathrm{DFP2}}, \cdots, W_{\mathrm{DFP}N})^{\mathrm{T}} = (\boldsymbol{R}_{\mathrm{meas}}^{\mathrm{T}}, \boldsymbol{W}_{\mathrm{DFP}}^{\mathrm{T}})^{\mathrm{T}}$.

Covariance matrix $\boldsymbol{U}_{W,\boldsymbol{a}}$ according to equation (11) is according to [7], paragraph 6.3.1

$$\boldsymbol{U}_{W,\boldsymbol{a}} = \boldsymbol{C}\, \boldsymbol{U}_{\boldsymbol{R}_{\mathrm{meas}},\boldsymbol{W}_{\mathrm{DFP}}}\, \boldsymbol{C}^{\mathrm{T}} \tag{12}$$

Matrix $\boldsymbol{C}$=-$\boldsymbol{C}_y \boldsymbol{C}_x$, where matrices of sensitive coefficients for model (11) are in form (see also [19], equations (12) and (13)) if we more consider that $1 - \sum_{i=1}^{N} a_i \frac{\partial f_i(W)}{\partial W} \approx 1$ then

$$\boldsymbol{C}_x = \begin{pmatrix} -\dfrac{1}{R_{\mathrm{TPW}}} & \dfrac{R}{R_{\mathrm{TPW}}^2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \tag{13}$$

and

$$\boldsymbol{C}_y = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & f_1(W_{\mathrm{DFP1}}) & f_2(W_{\mathrm{DFP1}}) & \cdots & f_N(W_{\mathrm{DFP1}}) \\ 0 & f_1(W_{\mathrm{DFP2}}) & f_2(W_{\mathrm{DFP2}}) & \cdots & f_N(W_{\mathrm{DFP2}}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & f_1(W_{\mathrm{DFP}N}) & f_2(W_{\mathrm{DFP}N}) & \cdots & f_N(W_{\mathrm{DFP}N}) \end{pmatrix} \tag{14}$$

For determination of covariance matrix $\boldsymbol{U}_{\boldsymbol{R}_{\mathrm{meas}},\boldsymbol{W}_{\mathrm{DFP}}}$ in the equation (13) assuming model

$$\begin{pmatrix} R_{\mathrm{out}} \\ R_{\mathrm{TPW\ out}} \\ W_{\mathrm{DFP1}} \\ W_{\mathrm{DFP2}} \\ \vdots \\ W_{\mathrm{DFP}N} \end{pmatrix} = \begin{pmatrix} R_{\mathrm{in}} \\ R_{\mathrm{TPW\ in}} \\ \dfrac{R_{\mathrm{DFP1}}}{R_{\mathrm{TPW1}}} \\ \dfrac{R_{\mathrm{DFP2}}}{R_{\mathrm{TPW2}}} \\ \vdots \\ \dfrac{R_{\mathrm{DFP}N}}{R_{\mathrm{TPW}N}} \end{pmatrix} \tag{16}$$

then

$$\boldsymbol{U}_{\boldsymbol{R}_{\mathrm{meas}},\boldsymbol{W}_{\mathrm{DFP}}} = \boldsymbol{C}\, \boldsymbol{U}_{\boldsymbol{R}_{\mathrm{meas\ in}},\boldsymbol{R}_{\mathrm{cal}}}\, \boldsymbol{C}^{\mathrm{T}} \tag{17}$$

Where matrix $\boldsymbol{C}$ is

$$\boldsymbol{C}=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{1}{R_{\mathrm{TPW1}}} & \cdots & 0 & -\dfrac{R_{\mathrm{DFP1}}}{R_{\mathrm{TPW1}}^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \dfrac{1}{R_{\mathrm{TPW}\,N}} & 0 & \cdots & -\dfrac{R_{\mathrm{DFP}N}}{R_{\mathrm{TPW\,N}}^2} \end{pmatrix} \quad (18)$$

Covariance matrix

$$\boldsymbol{U}_{\boldsymbol{R}\mathrm{meas\ in},\,\boldsymbol{R}\mathrm{cal}}=\begin{pmatrix} u^2(R) & u(R,R_{\mathrm{TPW}}) & u(R,R_{\mathrm{DFP1}}) & \cdots & u(R,R_{\mathrm{TPW}N}) \\ u(R_{\mathrm{TPW}},R) & u^2(R_{\mathrm{TPW}}) & u(R_{\mathrm{TPW}},R_{\mathrm{DFP1}}) & \cdots & u(R_{\mathrm{TPW}},R_{\mathrm{TPW}\,N}) \\ u(R_{\mathrm{DFP1}},R) & u(R_{\mathrm{DFP1}},R_{\mathrm{TPW}}) & u^2(R_{\mathrm{DFP1}}) & \cdots & u(R_{\mathrm{DFP1}},R_{\mathrm{TPW}N}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u(R_{\mathrm{TPW}N},R) & u(R_{\mathrm{TPW}N},R_{\mathrm{TPW}}) & u(R_{\mathrm{TPW}N},R_{\mathrm{DFP1}}) & \cdots & u^2(R_{\mathrm{TPW}N}) \end{pmatrix} \quad (20)$$

It has on the diagonal squares of SPRT resistances from temperature measurement and from SPRT calibration at DFPs. Corresponding covariances are outside the diagonal.

We can also write

$$\boldsymbol{U}_{\boldsymbol{R}\mathrm{meas\ in},\,\boldsymbol{R}\mathrm{cal}}=\begin{pmatrix} \boldsymbol{U}_{\boldsymbol{R}_{\mathrm{meas}}} & \boldsymbol{U}(\boldsymbol{R}_{\mathrm{meas\ in}},\boldsymbol{R}_{\mathrm{cal}}) \\ \boldsymbol{U}(\boldsymbol{R}_{\mathrm{meas\ in}},\boldsymbol{R}_{\mathrm{cal}})^{\mathrm{T}} & \boldsymbol{U}_{\boldsymbol{R}_{\mathrm{cal}}} \end{pmatrix}. \quad (21)$$

Here the covariance matrix $\boldsymbol{U}_{\boldsymbol{R}_{\mathrm{meas\ in}}}$ of type 2×2 shows uncertainties of SPRT resistances measurement at temperature T and at TPW in temperature measurement and covariances between them. Covariance matrix $\boldsymbol{U}_{\boldsymbol{R}\mathrm{cal}}$ of type $2N\times 2N$ shows uncertainties of SPRT resistances at DFPs and covariances between them from SPRT calibration (equation (5)). Covariance matrix $\boldsymbol{U}(\boldsymbol{R}_{\mathrm{measin}},\boldsymbol{R}_{\mathrm{cal}})$ of type $2\times 2N$ shows covariances between SPRT resistance measurements in temperature measurements and SPRT calibration.

Instead of covariance matrices is often preferable to consider the correlation matrices.

Effect of correlations between the SPRT resistances at DFPs for temperature subrange TPW-aluminum is shown in Fig. 1.

Different correlation coefficients between the SPRT resistances at DFPs when we use the SPRT outside the calibration laboratory are taken into account in Fig. 1. Correlations are $r(R_i,R_j)$ = 0,2, $u(R_i\,,R_{\mathrm{TPW}})$ = 0,3 and $r(R_i,R_j)$ = 0,3, $u(R_i\,,R_{\mathrm{TPW}})$ = 0,4. Uncertainty of SPRT resistance at TPW is 0,02 mΩ. Uncertainty $u(R)$ of resistance R and correlation $u(R,\,R_{\mathrm{TPW}})$ between the resistance of the client and the resistance at TPW are not considered.

On the basis of Fig.1 can be assumed that the covariances between SPRT resistances at the DFPs may affect uncertainty of temperature. Their contribution may be larger than the contribution of covariances between resistance ratios caused by using the same value of SPRT resistance at TPW in calibration and measurement. Considered correlations can occur in real situations.

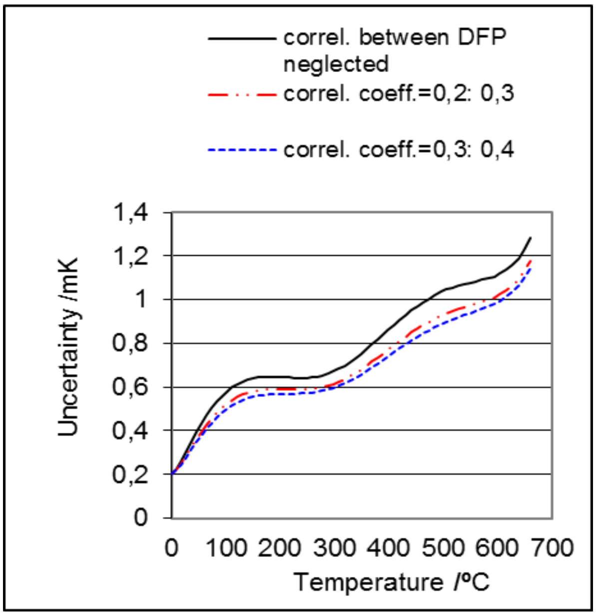

Figure 1. Uncertainty propagation of calibration for the subrange from TPW to Al.

3. CONCLUSION

This paper presents the determination of temperature uncertainty by means of calibrated SPRT at defining fixed points using matrix algebra. This approach is in accordance with the Supplement 2 to the GUM [7]. Contrary to Rosenkranz [19] this approach is based not on resistance ratios but on resistances obtained directly from calibration and measurement. It allows a very simple way to include in the calculation any correlations between SPRT resistances from measurement and calibration.

For this procedure is sufficient to determine the covariance matrix of resistances and we get on the output the resulting uncertainty of temperature.

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