XX IMEKO World Congress
Metrology for Green Growth
September 9–14, 2012, Busan, Republic of Korea

OPTIMIZATION OF CALIBRAION INTERVAL AND ADJUSTMENT LIMIT OF MESUREMENT DEVICE WITH STOCHASTICALLY DIFFUSIVE PROPERTY

K. Shirono/Presenter, H. Tanaka, M. Shiro and K. Ehara

National Metrology Institute of Japan (NMIJ), Tsukuba, Ibaragi, Japan, k.shirono@aist.go.jp

Abstract: A calibration and an adjustment of measurement device are keys to realize a metrological traceability. (Calibration means just checking without adjustment in this paper.) Although the traditional online quality engineering can be one of applications to determine calibration interval and adjustment limit in terms of cost, approximations used in the traditional quality engineering are too rough to take stochastic behaviour of a bias in a device into consideration. In this paper, we propose an optimum calibration interval and an optimum adjustment limit for a measurement device with a bias in accordance with Brownian dynamics concerning a long-term cost.

Keywords: Online quality engineering, Brownian dynamics, Stochastic differential equation, Calibration interval, Adjustment limit.

1. INTRODUCTION

A calibration and an adjustment of measurement device are keys to realize a metrological traceability. In the present study, calibration means just checking without adjustment as shown in 2.39-NOTE 3 in the definition of calibration in VIM3 [2]. And adjustment includes both of numerical and mechanical correction of a gap between calibration value and indication value. To realize a reliable calibration, in Japan, we have accreditation bodies to review calibration laboratories based on ISO/IEC 17025[1]. In the accreditations, calibration intervals are usually determined in a customary way and adjustments of measurement devices are basically considered to be done after every calibration. In terms of cost, this is not an optimum measurement control. We look at this calibration-adjustment issue from the view point of long term cost and propose a procedure to compute an optimum set of a calibration interval and an adjustment limit. Although, in reality, this issue has been discussed in the traditional online quality engineering on production field [3], the discussion is insufficient in the case where property varies stochastically. In this study, we propose optimized control when a bias in a measurement device behaves as Brownian motion, using parabolic loss function, which is usually employed in the traditional quality engineering.

2. NOMENCLATURE

In the present paper, the symbols below are basically used without explanations.

t: Time elapsed after a last adjustment.
T: Time interval between calibrations.
D: Adjustment limit.
$x(t)$: Bias of measurement device.
i: Number of calibrations after the last adjustment.
$y(iT)$: Measured value of $x(t)$ in the calibration of $t = iT$.
e_c: Measurement error in calibrations ($y(iT) = x(iT) + e_c$).
e_a: Error in an adjustment except for e_c ($x(0) = e_a - e_c$).
$y_t(t)$: Expanded function of $y(iT)$ with respect to continuous value of t ($y_t(t) = x(t) + e_m$).
σ_c: Population standard deviation of e_c.
σ_a: Population standard deviation of e_a.
$B(t)$: Normalized Brownian motion ($E_B[(B(t)\text{-}B(0))^2] = t$).
α: Diffusion coefficient.
ξ: Dimensionless adjustment limit equal to D^2/α^2T.
η: Dimensionless standard deviation of adjustment error equal to σ_a/D.
$E_z[.]$: Operator for expectation with respect to variable z.
$E[.]$: Operator for expectation with respect to B and e_a. ($E[.] = E_B[E_{ea}[.]]$).
k: Value of i sufficing $|y(iT)| > D$.
$\boldsymbol{y}$: Set of calibration values during $t = 0 \sim kT$ ($\boldsymbol{y} = (y(0), y(T), \ldots, y(kT))$).
C_Q: Loss (= cost for poor quality) during $t = 0 \sim kT$.
C_c: Cost for a single calibration.
C_a: Cost for a single adjustment.
C_T: Total cost during $t = 0 \sim kT$.
L: Expected cost per unit time.
T^*: Optimum calibration interval.
D^*: Optimum adjustment limit.
T': Optimum calibration interval in the traditional quality engineering.
D': Optimum adjustment limit in the traditional quality engineering.

3. FORMULATION

It is assumed that a loss per time is given by a parabolic function. This implies that a loss during $t = 0 \sim kT$, C_Q, is given as follows:

$$C_Q = \int_0^{kT} qx^2 dt \,. \qquad (1)$$

See reference 3 for determination of q. $x(0)$ is a bias from an aimed value just after an adjustment. Since an adjustment is done with a calibration result, an error in the calibration can influence an error in the adjustment. We divided an error in adjustment into two terms; a calibration error, e_c, and a total error of the other factors, e_a. Hence $x(0) = e_a - e_c$.

It is also assumed that motion of $x(t)$ is Brownian. Thus, the variation of $x(t)$, dx, can be expressed as

$$dx = \alpha dB \,. \qquad (2)$$

$y(iT)$ is the measured value of $x(iT)$ with the error of e_c. Here, e_c is assumed to be a constant. This means that a calibration error is basically caused by systematic effects. In other words, random effects in a calibration error are negligible. Since a pertinent measurement system control can reduce random effects in a calibration error, this premise is actually natural.

Since the value of $x(iT)$ is unknown eventually, $y(iT)$ is compared to an adjustment limit, D. When $|y(kT)| > D$, an adjustment is done. k does not depend on e_c but only on a Brownian motion, B. A total cost during $t = 0 \sim kT$, C_T, is given as follows:

$$C_T = \int_0^{kT} qx^2 dt + C_a + kC_c \,. \qquad (3)$$

Here a continuous function of $y_t(t) = x(t) + e_c$ is defined for convenience, because $y(t)$ is a discrete function which can be defined only when $t = iT$ (i = 1, 2, …, k). The following equation can be derived using $y_t(t)$:

$$\begin{aligned}\int_0^{kT} x^2 dt &= \int_0^{kT} (y_t - e_c)^2 dt \\ &= \int_0^{kT} y_t^{\,2} dt - 2e_c \int_0^{kT} y_t dt + \int_0^{kT} e_c^{\,2} dt\end{aligned} \,. \qquad (4)$$

This cannot be computed because we cannot know the value of e_c. Neither can Eq. (3) because Eq. (4) is a part of Eq. (3). However, it is possible to compute the expectation of Eq. (3) by regarding e_c as a random variable. This means that e_c is a constant with respect to time as assumed above but the constant value is derived stochastically. The expectation of the second term in the right hand side of Eq. (4) with respect to e_c equals 0 and that of the third term is $q\sigma_c^2 kT$. Based on the fact that k is irrelevant to e_c, we can obtain the expectation of C_T with respect to e_c, $E_{ec}[C_T]$, as follows:

$$E_{e_c}[C_T] = \int_0^{kT} qy_t^{\,2} dt + q\sigma_c^{\,2} kT + C_a + kC_c \,. \qquad (5)$$

We can compute the expectation of C_T after computing the expectation with respect to e_a and B as well as e_c. Let $E[.] \equiv E_B[E_{ea}[.]]$.

A time-weighted total cost per time, L, is a ratio of the expectation of the total cost, $E[C_T]$, to the expectation of time between adjustments, $E[kT]$. Since kT is irrelevant to e_c, $E_{ec}[E[kT]] = E[kT]$. Thus, L is given as follows:

$$\begin{aligned}L &= \frac{E\left[E_{e_m}[C_T]\right]}{E[kT]} \\ &= \frac{E\left[\int_0^{kT} qy_t^{\,2} dt\right] + q\sigma_m^{\,2} E[kT] + C_a + E[k]C_m}{E[kT]} \,. \\ &= \frac{E\left[\int_0^{kT} qy_t^{\,2} dt\right]}{E[kT]} + q\sigma_m^{\,2} + \frac{C_a}{E[kT]} + \frac{C_m}{T}\end{aligned} \qquad (6)$$

An optimum time calibration interval, T^*, and an optimum adjustment limit, D^*, can be defined as follows:

$$(T^*, D^*) = \underset{T,D}{\arg\min}(L) \,. \qquad (7)$$

The formulation is quite different from the one in the traditional quality engineering (See 5.3). Moreover, Section 7 provides a comment on a time lag.

4. COMPUTATION

Except for particular cases (see Section 5), the computation of Eq. (7) needs iterative calculations of Eq. (6). Eq. (6) is able to be computed numerically. However, if integration computations involving expectations are done directly in every iteration, that can be too time-consuming to finish the calculation in a realistic time. Here we propose some artifices to be computed.

In the calculation of $E[\int_0^{kT} qy_t^2 dt]$, we do not recommend to reproduce a Brownian motion, B, with short time interval as $B(0)$, $B(\Delta t)$, $B(2\Delta t)$, …, $B(K\Delta t = kT)$, but to reproduce a set of calibration value, $\boldsymbol{y} = (y(0), y(T), \ldots, y(kT))$. Even when $\boldsymbol{y}$ is given, a single Brownian motion is not determined and several B are possible to realize the pass of $\boldsymbol{y}$. Therefore, it is necessary to integrate $\int_0^{kT} qy_t^2 dt$ over all the possible Brownian motions weighting their probabilities. Here, $y_t(t)$ ($iT < t < (i+1)T$) is considered random variable which derived from a normal distribution with the mean of the weighted mean of $y(iT)$ and $y((i+1)T)$. The following equation can be derived:

$$E\left[\int_0^{kT} {y_t}^2 dt\right] = E_{e_a}\left[E_y\left[\sum_{i=0}^{k-1}\int_{iT_{cal}}^{(i+1)T} {y_t}^2 dt\right]\right]$$

$$E_{e_a}\left[E_y\left[\sum_{i=0}^{k-1}\left\{\frac{T}{3}\begin{pmatrix} y(iT)^2 + y((i+1)T)^2 \\ + y(iT)y((i+1)T)\end{pmatrix} + \frac{\alpha^2}{6}T^2\right\}\right]\right] \quad (8)$$

Since $E[kT] = E_{ea}[E_y[kT]]$, it is possible to calculate $E[\int_0^{kT} q{y_t}^2 dt]$ and $E[kT]$ in Eq. (6) over e_a and $\boldsymbol{y}$ with Monte Carlo integration method.

However, it is unrealistic to execute the computation of Eq. (6) every time T and D change in the iteration calculation to get a solution of Eq. (7). Therefore, it is proposed to regard L as the function of not only T and D but also a dimensionless parameter of $\xi = D^2/\alpha^2 T$ and $\eta = \sigma_a/D$. This is because $E[\int_0^{kT} q{y_t}^2 dt]$ and $E[kT]$ are proportional to the functions with the parameters of ξ and η. Thus,

$$E\left[\int_0^{kT} q{y_t}^2 dt\right] = \frac{qD^4}{\alpha^2} f(\xi,\eta), \quad (9)$$

$$E[kT] = \frac{D^2}{\alpha^2} g(\xi,\eta). \quad (10)$$

Concerning σ_a/D, $D^* \le \sigma_a$ is obvious because hunting oscillation was observed if not so. Therefore, it is adequate to focus on the cases only when $\eta \le 1$ when an optimization is the purpose of the calculation. Once $f(\xi,\eta)$ and $g(\xi,\eta)$ can be approximated in simple function of ξ and η, the calculation of Eq. (6) can be done easily. To approximate $f(\xi,\eta)$ and $g(\xi,\eta)$, we propose the following procedure.

(i) Set ξ and η. $T = 1/\xi$. $n = 0$.
(ii) $n = n + 1$. $y(0) = R_0$, where R_0 is derived from the normal distribution with the mean of 0 and the variance η.
(iii) For i, $y(iT) = y((i-1)T) + \sqrt{T}\cdot R$ until $|y(iT)| > 1$, where R is derived from the normal distribution with the mean of 0 and the variance 1.
(iv) $F_n = \sum_{i=0}^{k-1}\left\{\frac{T}{3}\begin{pmatrix} y(iT)^2 + y((i+1)T)^2 \\ + y(iT)y((i+1)T)\end{pmatrix} + \frac{\alpha^2}{6}T^2\right\}$ and $G_n = kT$. Go to (ii) until $N = n$.
(v) $f(\xi,\eta) = \frac{1}{N}\sum_{n=1}^{N} F_n$ and $g(\xi,\eta) = \frac{1}{N}\sum_{n=1}^{N} G_n$.

After calculating $f(\xi, \eta)$ and $g(\xi, \eta)$ for various combinations of (ξ, η), they are tabled or approximated to simple functions. The table or approximation functions together with Eqs. (9) and (10) provide total loss, L, in following equation without explicit integration calculations:

$$L = \frac{qD^2 f(\xi,\eta)}{g(\xi,\eta)} + q{\sigma_m}^2 + \frac{\alpha^2 C_a}{D^2 g(\xi)} + \frac{C_m}{T} \quad (11)$$

An example of the approximation functions for $f(\xi, \eta)$ and $g(\xi, \eta)$ applicable for $0.1 < \xi < 10$, $\eta < 1$ are shown in Appendix.

5. ANALYTICAL SOLUTIONS

The computation to realize the above concept has no difficulty other than the approximation of $f(\xi,\eta)$ and $g(\xi,\eta)$. However, there are the analytical solutions for several special cases and they are meaningful in terms of practical application.

5.1 Continuous calibration: When $C_c = 0$, continuous calibration ($T^* = 0$) is the solution for the calibration interval. Realistically, if one can calibrate a measurement device every day or before every measurement irrespective to its cost, the solution can be useful.

To start with, $\sigma_a = 0$ is assumed. In this case, the adjustment is done at the moment of $|x| = D$. And to determine D^*, $E[\int_0^{kT} q{y_t}^2 dt]$ and $E[kT]$ should be computed. These expectations can be calculated to solve equivalent differential equations [4]. Thus, L in Eq. (6) is given in the simple form:

$$L = q\frac{D^2}{6} + \alpha^2\frac{C_a}{D^2}. \quad (12)$$

Minimizing this, we obtain D^* as follows:

$$D^* = \left(\frac{6\alpha^2 C_a}{q}\right)^{1/4}. \quad (13)$$

When $\sigma_a > 0$ but $D^* >> \sigma_a$, D^* is given approximately as follows:

$$D^* = \left[\left(\frac{6\alpha^2 C_a}{q}\right)^{1/2} + {\sigma_a}^2\right]^{1/2}. \quad (14)$$

In this case, $E[kT] = (D^{*2} - {\sigma_a}^2)/\alpha^2$ approximately. Recommended procedure is follows: (i) Calculate D^* in accordance with Eq. (13). (ii) Confirm $\sigma_a < 1/3\cdot D^*$. (iii) Confirm $T < 1/10\cdot(D^{*2} - {\sigma_a}^2)/\alpha^2$. If $\sigma_a \ge 1/3\cdot D^*$ or/and $T \ge 1/10\cdot(D^2 - {\sigma_a}^2)/\alpha^2$, numerical calculation is recommended.

5.2 Adjustment after every calibration: When $C_a = 0$ and $\sigma_a = 0$, adjusting after every calibration is the best way ($D^* = 0$). Even when $C_a > 0$, if device is adjusted after every calibration, k is not a random variable but a constant of $k = 1$. The expectations in Eq. (6) can be calculated as well as Subsec. 5.1, and L can be given as follows:

$$L = \frac{C_c + C_a}{T} + \frac{\alpha^2}{2}qT. \quad (15)$$

Minimizing this, we obtain T^* as follows:

$$T^* = \left(\frac{2(C_c + C_a)}{\alpha^2 q} \right)^{1/2} \qquad (16)$$

When $\sigma_a > 0$, to prevent hunting oscillation, adjusting only when $|y(t)| > \sigma_a$ is the best way to control ($D^* = \sigma_a$). When σ_a^2 would be negligibly smaller than $\alpha^2 T$ as $\sigma_a^2/\alpha^2 T^* < 1/20$, T^* is given by Eq. (16) approximately. It should be noted that since σ_a^2 does not contain the component of σ_c^2, σ_a^2 can be 0 when adjustment is conducted numerically.

5.3 Comparison to the traditional online quality engineering's solution: In the traditional quality engineering, the adjustment limit, *D*, and the calibration interval, *T*, are discussed. And the optimized combination is given as (*D'*, *T'*) in following equations:

$$D' = \left(\frac{3\alpha^2 C_a}{q} \right)^{1/4} \qquad (17)$$

$$T' = \left(\frac{2C_c}{\alpha^2 q} \right)^{1/2} \qquad (18)$$

These equations resemble to Eqs. (13) and (16), respectively. The minute difference between Eq. (13) and (17) is ascribed to the deference of the probability density functions of *x*. In the traditional quality engineering, the probability density function of *x* is *a priori* given as the uniform distribution with the range of $(-D, D)$. On the other hand, we found that the density distribution of *x* in Subsec. 5.1 is (isosceles) triangle with the same range when $\sigma_c = 0$. Eqs. (16) and (18) are the same as each other when $C_a = 0$. However, it should be noted that D^* and T^* given in Subsec. 5.1 and Subsec. 5.2 are the optimized parameters in different situations. Thus, although the equations provided by the traditional quality engineering can be meaningful in some situations, those cannot be solutions for a bias in accordance with Brownian motion.

6. EXAMPLE

Our method is applied to an example in Subsec. 23.3 of "Taguchi's quality engineering handbook" (Wiley-Interscience) [3]. This example is not for an adjustment in calibration but for an adjustment in production, but the mathematical properties of these two issues are totally the same. Although in this example uncertainties are not taken into consideration, they are considered in the present study for a comparison.

The symbols are changed from the original ones to the symbols used in this paper. Let the parameters of (α, q, σ_c, σ_a, C_c, C_a) = (0.144 μm/(unit)$^{1/2}$, 0.003556 \$/unit, 0 μm, 0 μm, 1.5 \$, 12 \$) and this parameter set is referred to as Parameter set A. Moreover, Parameter sets B and C are given as (α, q, σ_c, σ_a, C_c, C_a) = (0.144 μm/(unit)$^{1/2}$, 0.003556 \$/unit, 0.2 μm, 0 μm, 1.5 \$, 12 \$) and (0.144 μm/(unit)$^{1/2}$, 0.003556 \$/unit, 0.2 μm, 0.5 μm, 1.5 \$, 12 \$) respectively to look at the influence of the uncertainties of calibration and adjustment. We just neglect the time-lag parameter for simplicity. See Sec. 7 regarding this.

To compare our proposal to the traditional online quality engineering, the result of Parameter set A is discussed. The optimum point based on the traditional online quality engineering is given as (*D'*, *T'*) = (3.8 μm, 201 units) as the optimum point. Figure 1 shows the evaluated cost per production unit. This is calculated based on Eq. (6). Figure 1 obviously shows that (*D'*, *T'*) is not the best point in terms of cost. (D^*, T^*) is evaluated to be (2.9 μm, 292 units). The minimized cost is *L* = \$ 0.0344/unit. Thus, it is concluded that traditional quality engineering is not applicable for the adjustment of measurement device with a bias diffusing as Brownian motion and the numerical approach is usually necessary to obtain the optimum parameters in that case.

For Parameter set B, the optimum point is (D^*, T^*) = (2.9 μm, 292 units) similar to Parameter set A, but the minimized cost is *L* = \$ 0.0346/unit and different from Parameter set A. It is natural that the difference in calibration uncertainty affect only the cost. On the other hand, Parameter set C gives the optimum point of (D^*, T^*) = (3.0 μm, 287 units) and different from that provided by Parameter set A. It is found that adjustment uncertainty affects the optimum point. Moreover, the minimized cost is *L* = \$ 0.0349/unit with Parameter set C. These results imply that uncertainty in adjustment is influential for the total cost to be larger as well as uncertainty in calibration.

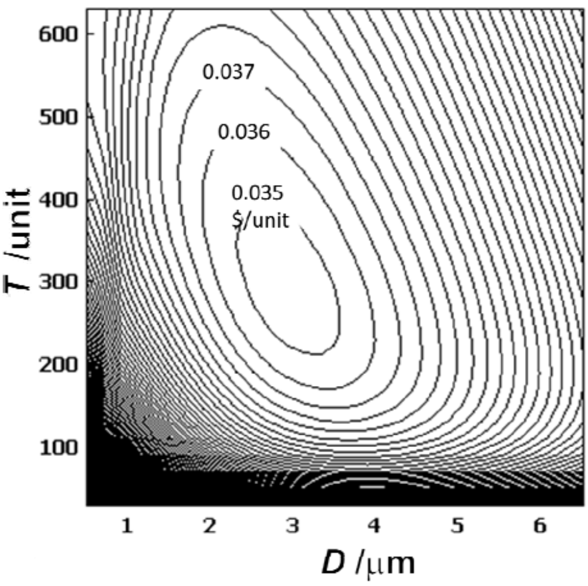

Fig. 1 The contour of the evaluated cost per production unit for Parameter set A. *x*-axis shows an adjustment limit and *y*-axis shows calibration interval. The contours are drawn every \$ 0.001/unit from \$ 0.035/unit. The minimum expected cost per unit time is *L* = \$ 0.0344/unit given at (D^*, T^*) = (2.9 μm, 292 units), which differs from the optimum point calculated using the traditional online quality engineering of (*D'*, *T'*) = (3.8 μm, 201 units).

7. SUMAMRY

The optimized calibration interval and adjustment limit are not given based on the traditional online quality engineering when a bias of measurement device diffuses in accordance with Brownian motion. We proposed the method to optimize the parameters based on the mathematical formalism of the long-term-cost including the effect of uncertainties of a calibration and an adjustment. We apply this method to a realistic example and confirm the differences between the results of the traditional quality engineering and the proposed method.

The time lag parameter, ΔT, is not handled in the present study. However the extension is easy. Formulation can be done to replace $E[\int_0^{kT} q y_t^2 dt]$ and $E[kT]$ in Eq. (6) to $E[\int_0^{kT} q y_t^2 dt + y_t(kT)^2 \Delta T + \alpha \Delta T^2/2]$ and $E[kT] + \Delta T$ respectively.

In reality, the analytical solutions proposed in Sec. 6 will be useful in actual systems. The conditioning for an application of the analytical solutions will be our feature task.

References:

[1] ISO 17025, "General requirements for the competence of testing and calibration laboratories", 2005.

[2] JCGM (BIPM, IEC, IFCC,ILAC, ISO, IUPAC, IUPAP and OIML),."International vocabulary of metrology — Basic and general concepts and associated terms (VIM)", 2008.

[3] G. Taguchi S. Chowdhury, Y. Wu, "Taguchi's Quality Engineering handbook", John Wiley & sons, Inc., Hoboken, New Jersey.

[4] B. K. Øksendal, "Stochastic Differential Equations: An Introduction With Applications" in Japanese, Springer Japan, Tokyo, 1999.

APPENDIX

Here, the approximation equations for $f(\xi,\eta)$ and $g(\xi,\eta)$ in Sec. 4. Let $\xi = D^2/\alpha^2 T$, $\eta = \sigma_a/D$ and these equations can be applied when $0.1 < \xi < 10$ and $\eta < 1$. Letting $\zeta = \ln(D^2/\alpha^2 T)$, the approximation equations are follows:

$$\ln(f(\xi,\eta)) = f(\zeta, \eta)$$
$$= 0.1(10.3+0.152\eta - 3.445\eta^2+1.94\eta^3+ - 0.415\eta^4)$$
$$+ 0.1(-4.79+0.265\eta - 3.31\eta^2+2.43\eta^3 - 0.642\eta^4)\zeta$$
$$+ 0.01(8.65+0.630\eta - 5.06\eta^2+1.70\eta^3+0.923\eta^4)\zeta^2$$
$$+ 0.001(-1.01 - 0.437\eta+5.66\eta^2-1.61\eta^3 - 0.8545\eta^4)\zeta^3$$
$$+ 0.001(-1.08 - 0.525\eta+3.07\eta^2-2.79\eta^3+0.785\eta^4)\zeta^4$$

$$\ln(g(\xi,\eta)) = \gamma(\zeta,\eta)$$
$$= 0.1(6.80+0.133\eta+1.47\eta^2+1.50\eta^3 - 0.402\eta^4)$$
$$+ (-1.14+0.0226\eta - 0.160\eta^2+0.0317\eta^3+0.0555\eta^4)\zeta$$
$$+ 0.1(1.84+0.141\eta - 1.17\eta^2+0.779\eta^3 - 0.275\eta^4)\zeta^2$$
$$+ 0.001(1.79 - 1.84\eta+14.5\eta^2 - 22.3\eta^3+9.57\eta^4)\zeta^3$$
$$+ 0.001(-2.45 - 0.514\eta+4.39\eta^2 - 3.76\eta^3+1.41\eta^4)\zeta^4$$