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THE PLANE ANGLE CONCEPT AND ITS UNIT IN THE CONTEXT OF TRACEABILITY PROBLEM

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Abstract: The features of traceability structure of plane angle unit are considered. It is shown that the structure is contradicted the legalized unit, *radian*. It is offered to remove the contradiction by institution of the full angle (revolution) as the unit. The set of the problems demanding decisions for realization of this offer is outlined.

Keywords: angle unit, reference standard, traceability.

1. INTRODUCTION

The standard structure of unit transfer from the primary reference standard to subordinate, on accuracy, standards and working measuring instruments looks like:

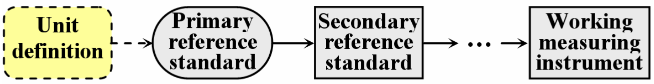

Fig. 1. Usual structure of unit transfer (traceability)

According to decisions of The XIth (1960) and The XIXth (1989) CGPM, legislatively established unit of a plane angle, *radian*, is defined in such a way: *'the radian is the plane angle between two radii of a circle, which cut off on the circumference an arc equal in length to the radius'*. Alongside with a radian, it is supposed to use traditional units, *degree* / *minute* / *second*, without any restrictions [1]. The hierarchy of these units is shown in Fig. 2. For the succeeding composition, the full angle (revolution) is also added in this scheme.

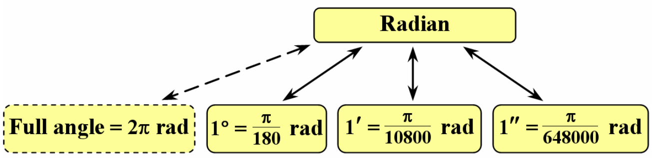

Fig. 2. Prescribed hierarchy scheme of angle units

However, angular unit, which is defined in a similar way, has obviously "non-metrical" nature. Really, anybody does not make the goniometric devices based on measurement of *length* of the corresponding arch and its radius with succeeding calculation of their ratio. The evident reason is that expenses for manufacturing of the similar device (which is closed, by the principle of operation, to coordinate measuring machines) would be rather significant by virtue of a difficult problem of exact "overlapping" the ends of the basic arch and two radiuses forming the angle. Therefore final accuracy would appear at the best on an average level.

2. ANGLE UNIT TRACEABILITY PROBLEM

The sphere of angular measurements is noted for that there is *the natural etalon*, namely, the full angle. It is directly used in measurements; moreover, it is the base for design of all precision angular measuring instruments including reference standards of different levels. Thus, the structure of transfer of plane angle unit essentially differs from the standard one (Fig. 1) and has the following form:

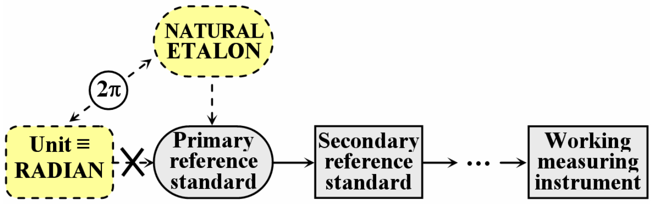

Fig. 3. Real structure of angle unit traceability

Actually the scheme implies a presence of *the conceptual contradiction*: the breakaway from legalized unit of a plane angle, radian, arises already at the uppermost accuracy level. This statement follows from the symbolic relation outlined in Fig. 4, where it is shown that the specification of the primary reference standard of a plane angle is based on inextricably linked with concepts of the full angle and procedure of its division.

Fig. 4. Principle of circular scale division

Moreover, procedure of division of the full angle underlies practically all goniometric devices [2].

Coming back to radian, it is necessary to notice, that the nature of this concept is of mathematical only. The valid definition of radian is properly equivalent to expression «to measure length of an arch in radii». If to analyze this expression, it becomes clear that the case in point is «mathematical measurement» but not measurement in strict metrological sense.

Thus the radius is unit of zoom, no more than that. The choice of such zoom is dictated exclusively by needs of mathematics to maintenance the known ratio, $\sin x \approx x$, which simplifies some formula records. That is the reason why Leonard Euler, in XVIII century, has brought this angular unit in daily mathematical use but not in metrical one. Thus, the radian is unit extremely in mathematical sense, setting some angular zoom.

3. THE OPERATIONAL APPROACH AS THE WAY TO SOLVE THE PROBLEM

The contradiction between principles of real angular measurements [3], including unit transfer, and formal definition of the unit can be taken off within the framework of the operational approach [4].

It is well known that the operational approach involves generally the institution of *five rules*, to which a quantity M has to be fulfilled to be measurable one. In other words, the case in point is the rules permitting to characterize an attribute of the object $A \in \mathsf{A}$ using compound number $M = M(A)$ so that we could collate numbers instead of attributes (accordingly, objects)[1]. In brief the following rules for measurable quantity introduction are established.

The equivalence rule 1 characterizes the empirical relation of equality E for the quantity M:

$$\text{if } E_M(A, B) \text{ then } M(A) = M(B). \qquad (1)$$

It means that, if relation E_M for objects A and B holds, the objects will be characterized by the same proportions (intensity) of the quantity M.

The ordering rule 2 introduces corresponding relation L_M for the objects A and B:

$$\text{if } L_M(A, B) \text{ then } M(A) < M(B). \qquad (2)$$

In other words, if relation L_M exists for the objects A and B so the objects are defined by different proportions (intensities) of the quantity M with the proportions (intensity) for A is smaller then for B. It means that the set of objects A can be strictly ordered by means of mapping $M: \mathsf{A} \to \mathsf{M}$, where $\mathsf{M} = \{M(A)\} \subset \mathsf{R}$:

$$\text{for } [\forall A, B \in \mathsf{A} \text{ not } E_M(A, B)] \\ \text{either } L_M(A, B) \text{ or } L_M(B, A). \qquad (3)$$

The zero rule 3 is directed to establishment of the first special value of the quantity M:

$$\exists\, A_0 \in \mathsf{A}: \quad M(A_0) = M_0\,, \qquad (4)$$

which sets the readout zero of the quantity M. In most practical cases, $M_0 = 0$. The mentioned value specifies an object or its particular condition, which has to be reliably reproducible and easily identifiable.

The unity rule 4 establishes the second special value of the quantity M:

$$\exists\, A_1 \in \mathsf{A}: \quad M(A_1) = M_1 \qquad (5)$$

Formally, the rule 4 coincides with the rule 3. But zero value M_0 is usually connected with that is no manifestation of the measurement object attribute which is expressed by the quantity M. By contrast, the second value M_1 is selected knowingly for measurement. Two allocated values are the base for establishment of quantity unity (measurement unity). Often and often, the second special value just represents the unity: $M_1 = 1$ or it is multiple unit.

The differences rule 5 determines the scale of the quantity M (measurement scale) assigning conditions ED_M for the equality E of two differences D of proportions (intensities) of the value M:

$$\text{if } ED_M(A, B, C, D) \\ \text{then } M(A) - M(B) = M(C) - M(D). \qquad (6)$$

It means that if empirical conditions ED_M exist for four objects, then it can be assert that the difference D_M between proportions (intensities) of the value M for any two objects A and B is the same one as for C and D.

It has be noted, once more, that the concept "different objects A and B" being used above is included, as particular case, in the notion "different state of A and B".

The above-mentioned set of rules became simpler if objects A and B are, in a sense, additive concerning inherent attribute which is expressed by the quantity M. It means that there is a possibility to join objects in such a way that the joined object $A \oplus B$ is characterized be the same value M but of total proportions (intensity):

$$M(A \oplus B) = M(A) + M(B). \qquad (7)$$

The above-mentioned additivity rule by essence replaces the rule 5 of the full set[2].

Above-mentioned rules application to plane angle is lightened if the plane angle is interpreted as the result of *rotation around the fixed axis* [5, 3]. It permits to refuse infinitely extensional objects which are present into the mathematical definition of angle as a part of plane.

It is proposed the following description of the object A as a carrier of attribute M. A natural model of a rotation angle α is circular sector S_α of any proper radius. The value of interest is preset by equation $M(S_\alpha) = \alpha$. Above-mentioned rules are formulated below for a plane angle, which is treated as rotation angle.

The equivalency rule (E) is established operationally for a plane angle by absolute angular matching two sectors S_α, S_β (i. e., their centres and forming boundary radii without respect to their radii relation):

[1] **A denotes the set of all objects or all different conditions of the object.**

[2] **More sophisticated reasoning permits to reduce the set of rules addressed to so kind objects to the following three rules: of equivalency, of additivity, and of unity.**

$$\text{if } E(S_\alpha, S_\beta) \text{ then } \alpha = \beta , \qquad (8)$$

what is illustrated in fig. 5.

The ordering rule (L) supposes particular angular matching sectors by matching their centres and two boundary radii – on one radius from every sector (with mutual overlap of the sectors):

$$\text{if } L(S_\alpha, S_\beta) \text{ then } \alpha < \beta , \qquad (9)$$

what is illustrated in fig. 6.

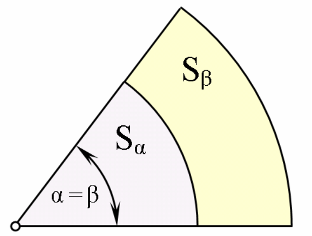

Fig. 5. The E-rule for sectors as plane angle's model

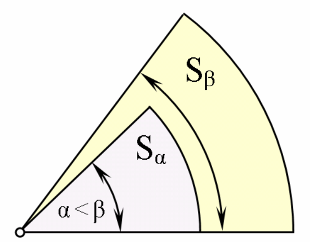

Fig. 6. The L-rule for sectors as plane angle's model

The zero rule (Z) is introduced on the base of rotation of any boundary radius of the sector in direction to the other radius up to their superposition: $\alpha_0 = 0$ (see fig. 7).

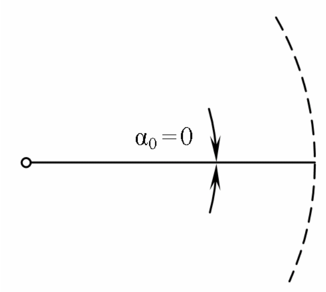

Fig. 7. The Z-rule for sectors as plane angle's model

The unity rule (U) is realised successfully with chose of *the full angle as the unit*: α_1 is equal of the full angle (one return), what will permit to give up the known mathematical definition of angle, which is not applicable for measurement. Such a choice of α_1 is to seem as natural one if to take into account that revolving radius will be cyclically superposed with the fixed radius in positions assigning the rotation angle which is multiple of full angle. It is illustrated in fig. 8.

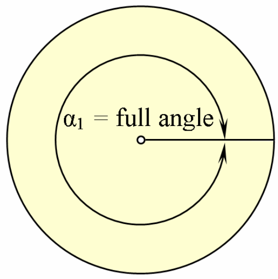

Fig. 8. The U-rule for sectors as plane angle's model

At last, the additivity rule (A) is instituted on the base of the simple procedure of objects banding. So, the combined sector $S_\alpha \oplus S_\beta$ is formed by matching centres of rotation for sectors S_α and S_β , and two boundary radii – one by one from every sector (without matching sectors). Evidently, its angle is equal $M(S_\alpha \oplus S_\beta) = \alpha + \beta = M(S_\alpha) + M(S_\beta)$ (see fig. 9).

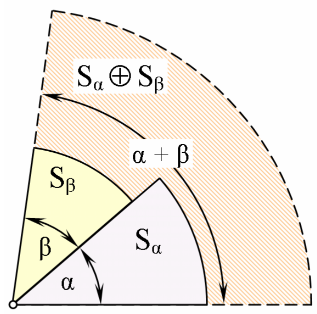

Fig. 9. The A-rule for sectors as plane angle's model

It should be remarked that generalisation of the sectors banding procedure on a case of $\alpha + \beta > \mathbf{360°}$ is made by additional counting full revolution number, and it is not a particular problem. In this case, the mathematical model is a sector on multisheeted surface.

So, the proposed concept permits to realise, on the base of "natural" unity introduction, the natural reference standard as the complete revolution (by rotation movement). As the result, the "metrical" approach to foundation construct of angle measurement can be actualised down-the-line.

The above-mentioned reasons permit to propose the following hierarchy for the plane angle units instead of the working one (see fig. 10).

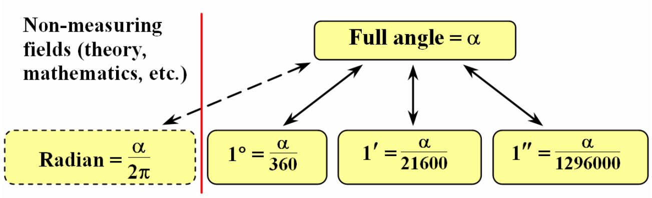

Fig. 10. Real hierarchy scheme of angle units

Legitimating the proposed hierarchy of the units will permit to bring their status to conformity with the procedures being made actually with reference standards. At that the submultiples units will be widely used as before in particular for working measuring instrument calibration. This entire permits to refuse of a use of infinitely lengthy objects which are present in mathematical definition of angle as a part of the plane. The natural model of rotation angle is circular sector of proper radius.

In other words, the case at hand is *an establishment of the full angle as unit* that will allow leaving the well-known mathematical definition of an angle, which is not suitable for the measurement purposes.

Giving of legality of the suggested unit hierarchy will allow bringing their status to accordance with measurement procedures, which are really carried out at reference standards. Thus traditional units of «support echelon» (degree / minute / second) will still be often used in practical measurements, including verification (graduation or calibration) of working measuring instruments.

As to dimension of above-mentioned portions of the unit, it is caused by dimension of a plane angle which, in turn, depends on the status of the suggested unit in the International System of Units [6] or, what is the same, the status of plane angle in system of physical values. There are serious arguments for the benefit of recognition of an angle to be stand-alone value, and accordingly, the full angle – stand-alone unit.

If to agree with that, dimension of an angle comes into *identity* and there is an opportunity to choose between two variants: (i) to consider the full angle as non-dimensional value which is expressed by abstract number "one"; (ii) to accept its dimensional and to express the concrete number, for example, “one revolution (turn)” [1]. In any case, the present ratio is saved:

$$\textbf{full angle} \equiv \textbf{360°}, \text{ etc.,} \tag{10}$$

and portions of the unit are used without changes, i. e., its sub-multiples are not decimal.

4. CONCLUSIONS

Introduction of the full angle as unit, alongside with "rotary" definition of an angle, will allow:

(i) to coordinate conceptually and methodologically, in the field of angular measurements, value and unit definitions with methods of reproduction and unit transfer, and also its realization in practical measurements and working measuring instruments;

(ii) to provide, on the basis of (i), realization of angular measurement unity at the expense of transition to measurements in legalized units;

(iii) to create a basis for decentralization of system of plane angle measurement unity;

(iv) to bring the methodological base under abundant differential angular measurements, in which self-checking of accuracy of plane angle division procedure is realized.

For realization of the specified proposal, it is required to solve the following problems:

(I) to define the status of plane angle unit in the International System of Units (derived, the basic or "over-system");

(II) to establish dimension of suggested unit (of revolution or non-dimensional);

(Ш) accordingly (II), to define dimensions of portions of the unit (degree / minute / second);

(IV) to establish restrictions and, accordingly, to formulate criterion of division procedure applicability on the basis of abundant differential measurements of angles as marks of circular scales starting from the following qualitative representation:

Result accuracy	~	Compared scales stability	+	Comparator sensitivity

The decision of the problems (I) – (Ш), apparently, will demand the analysis of the closed angular scale status in classification, which is accepted in the theory of measurement scales.

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