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TIME-VARYING BUTTERWORTH FILTERS WITH COMPENSATED GROUP DELAY

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Abstract: This paper presents the concept of time-varying Butterworth filters with linear phase response. The compensation of the phase characteristics is carried out with the aid of the phase shifter system. The paper shows that it is possible to shorten the transient state in low-pass phase-compensated analog filters by varying in time selected parameters. This paper contains simulation results of the proposed filters and comparison with the classic circuits.

Keywords: Butterworth filters, time-varying systems, group delay compensation.

1. INTRODUCTION

The fast evolution of the digital signal processing in the last years has conducted to the reducing of interest of new solutions in the analog technics. Nowadays it is necessary to improve the quality of the analog devices, especially in these places, where we cannot use the digital technics. There are many cases when it is important to preserve the sharpness of the step response and at the same time the phase linearity. The design methods of analog filters are described in detail in the rich literature [1], [2], [3], [4] and concern mainly filters with constant parameters.

The main goal of the Butterworth approximation is to obtain a maximally flat gain characteristics in the filter pass-band. The square of the gain characteristics of the low-pass Butterworth filter can be written as follows:

$$|G(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_0}\right)^{2N}} \tag{1}$$

where: N - order of the filter, ω_0 - 3dB limit frequency. Low-pass Butterworth filters are forming as a result of specified approximation of the gain characteristics, however the phase characteristics is the secondary effect of this approximation. In the result of this effect, the phase characteristics is nonlinear. Nonlinearity of the phase characteristics causes that the dynamic properties of designed filter may be undesirable.

2. GROUP DELAY COMPENSATION

For the purpose of the group delay compensation one can use the phase shifters [3], [5]. Transfer function of the first order phase shifter can be written as follows:

$$H_1(s) = \frac{1-\mu s}{1+\mu s} \tag{2}$$

but in the case of the second order phase shiter the transfer function has the following form:

$$H_2(s) = \frac{\left(\frac{s}{\omega_{0p}}\right)^2 - \frac{1}{2Q}\left(\frac{s}{\omega_{0p}}\right)+1}{\left(\frac{s}{\omega_{0p}}\right)^2 + \frac{1}{2Q}\left(\frac{s}{\omega_{0p}}\right)+1} \tag{3}$$

The above mentioned systems are stable when $\mu > 0$ and $Q > 0$. The specificquality of the phase shifters is its constancy of the gain characteristics. These systems are only used to form the phase characteristics. Owing to these properties we know that if we connect in series the phase shifter to the compensating filter then the gain characteristics of the resultant system remains without any changes, however the group delay characteristics will be the result from a summation of the phase shifter and the compensating filter characteristics. The procedure of the group delay compensation is based on the choice of the phase shifter(s) parameters in order that the resultant group delay characteristics will be as flat as possible in the filter pass-band.

The N-th order low-pass filter structure can be described by the transfer function which is the product of 2-nd order systems for even filter orders or the product of 2-nd order systems and one 1-st order system for odd filter orders. This can be written as follows:

$$G(s) = \frac{1}{\prod_i \left(\frac{1}{\omega_{0i}^2}s^2 + \frac{2\beta_i}{\omega_{0i}}s+1\right)} \tag{4}$$

for even orders $N = 2i$ and for odd filter orders $N = 2i + 1$ the transfer function has the following form:

$$G(s) = \frac{1}{(1+sT)\prod_i \left(\frac{1}{\omega_{0i}^2}s^2 + \frac{2\beta_i}{\omega_{0i}}s+1\right)} \tag{5}$$

In the case of the filter with possible linear phase the characteristic of the group delay $D(\omega)$ is very useful. The group delay is defined by the derivative of the phase characteristics $\varphi(\omega)$ with regard to the pulsation ω with the minus sign:

$$D(\omega) = \frac{d\varphi(\omega)}{d\omega} \qquad (6)$$

For the N-th order Butterworth filter the group delay can be expressed as follows:

$$D_N(\omega) = \frac{a_{N-1}\omega^{2(N-1)} + a_{N-2}\omega^{2(N-2)} + \ldots + a_1\omega^2 + a_0}{\omega^{2N} + 1} \qquad (7)$$

The Taylor expansion of the above mentioned function round about $\omega = 0$ has the following form:

$$D_G(\omega) \approx p_0 + p_1\omega^2 + p_3\omega^4 + \ldots \qquad (8)$$

The group delay of the second order phase shifter given by (3) has the following form:

$$D_{H_2}(\omega) = \frac{1}{\omega_{0p}} \frac{2Q\left(\frac{\omega}{\omega_{0p}}\right)^2 + 2Q}{Q^2\left(\frac{\omega}{\omega_{0p}}\right)^4 + \left(1-2Q^2\right)\left(\frac{\omega}{\omega_{0p}}\right)^2 + Q^2} \qquad (9)$$

The Taylor expansion of the group delay of the second order phase shifter round about $\omega = 0$ has the following form:

$$D_{H_2}(\omega) \approx \frac{2}{Q\omega_{0p}} + 2\cdot\frac{3Q^2-1}{Q^3\omega_{0p}{}^3}\omega^2 + 2\cdot\frac{5Q^4-5Q^2+1}{Q^5\omega_{0p}{}^5}\omega^4 + \ldots \qquad (10)$$

The sum of both group delay expansions can be written as follows:

$$D_G(\omega) + D_{H_2}(\omega) \approx \left(p_0 + \frac{2}{Q\omega_{0p}}\right) + \left(p_1 + 2\cdot\frac{3Q^2-1}{Q^3\omega_{0p}{}^3}\right)\omega^2 + \left(p_2 + 2\cdot\frac{5Q^4-5Q^2+1}{Q^5\omega_{0p}{}^5}\right)\omega^4 + \ldots \qquad (11)$$

Parameters Q and ω_{0p} should be selected in this way in order to eliminate the term of the second and fourth order of $D_G(\omega)+D_H(\omega)$. For this purpose it is necessary to solve a suitable system of equations as follows:

$$\begin{cases} p_1 + 2\cdot\frac{3Q^2-1}{Q^3\omega_{0p}{}^3} = 0 \\ p_2 + 2\cdot\frac{5Q^4-5Q^2+1}{Q^5\omega_{0p}{}^5} = 0 \end{cases} \qquad (12)$$

All important parameters of phase-compensated Butterworth filters from 3-rd up to 6-th order are presented in Table 1.

Table 1. Parameters of phase-compensated Butterworth filters from 3-rd up to 6-th order.

N	ω_{0p}	Q	t_u [s]	t_{uc} [s]
3	1.0232	0.5509	5.9654	7.0927
4	1.0955	0.5434	6.8523	9.5853
5	1.0813	0.5406	7.6572	10.5104
6	1.0521	0.5389	10.7680	11.3863

where t_u is the settling time of the original filter but t_{uc} is the settling time of the filter with equalized group delay response.

The comparison of group delay characteristics of the 4-th order original filter and the phase-compensated Butterworth filter is shown in figs. 1 and 2.

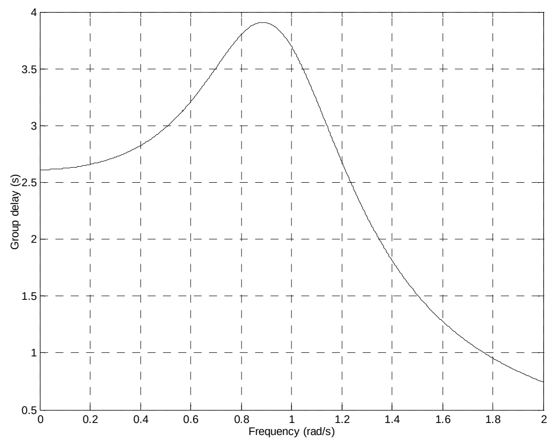

Fig. 1. Group delay response of the 4-th order original Butterworth filter.

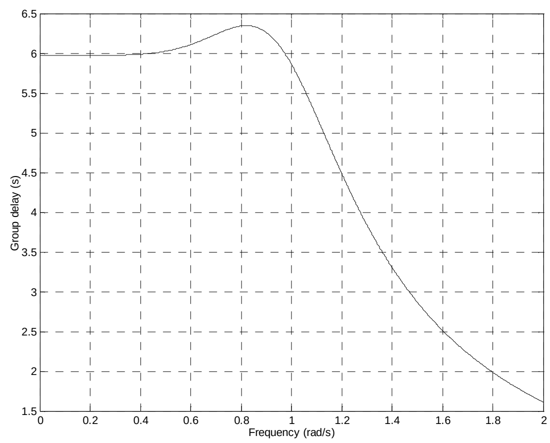

Fig. 2. Group delay response of the 4-th order phase-compensated Butterworth filter.

How one can notice, the compensation of the group delay characteristics brings good results, because compensated characteristics is considerably more flat in the filter pass-band than in the original Butterworth filter. The group delay value of the phase-compensated filter is obviously greater than in the original filter, however this value does not play a greater part. For the systems from impulse technique the important feature is constancy of the group delay in the filter pass-band. Compensation of the group delay was carried out at the cost of the extension of the filter transient state.

3. TIME-VARYING COEFFICIENTS

For constant parameter filters there are only small possibilities of shortening the transient state and this is because the filter parameters are calculated on the base of the assumed approximation method of the frequency characteristics (phase or gain) which guarantees that the frequency specifications are satisfied without taking into consideration the character of the transient state. The possibility of improvement of the filter properties is provided by varying in time their parameters. Analysis of parametric systems is much more complicated and the number of works on this subject is rather low. Only very specific types of parametric differential equations can be solved analytically. However the development of modern simulation techniques makes examination of parametric systems possible. The paper shows that it is possible to shorten the transient state in low-pass analog filters by varying in time selected parameters.

The indeterminacy principle says that it is not possible to achieve a shorter rise time of the low-pass filter output signal when the filter pass-band is constant. One can obtain significant changes of duration of the transient state by variation of filter band-pass in low-pass filters. This procedure is connected with the change of value of parameters ω_0, β and T which leads to non-realization of the imposed filter frequency characteristics.

Time varying filter design is the result of modeling of system of ordinary differential equations with varying coefficients described as follows:

$$\begin{cases} a_1(t)\dfrac{d^2y_1(t)}{dt^2}+b_1(t)\dfrac{dy_1(t)}{dt}+y_1(t)=k_1x(t) \\ a_2(t)\dfrac{d^2y_2(t)}{dt^2}+b_2(t)\dfrac{dy_2(t)}{dt}+y_2(t)=k_2y_1(t) \\ a_i(t)\dfrac{d^2y_i(t)}{dt^2}+b_i(t)\dfrac{dy_i(t)}{dt}+y_i(t)=k_iy_{i-1}(t) \end{cases} \tag{13}$$

where:

$$k_1\cdot k_2\cdot\ldots\cdot k_i=k_u,\ \ a_i(t)=\frac{1}{\omega_{0i}{}^2(t)},\ \ b_i(t)=\frac{2\beta_i(t)}{\omega_{oi}(t)}$$

In the case of odd filter order $a_1(t) = 0$ and $b_1(t) = T(t)$.

Dynamic properties of arbitrary low-pass filter can be described by means of the damping factor β_i, characteristic frequency ω_{0i} and time constant T (only in the case of odd filter orders).

Let's assume, that all those parameters belong to certain set:

$$F=\left\{\beta_i,\omega_{oi},\frac{1}{T}\right\} \tag{14}$$

Next, let's create the set $F(t)$ of filter coefficients functions:

$$F(t)=\left\{\beta_i(t),\omega_{oi}(t),\frac{1}{T}(t)\right\} \tag{15}$$

The main assumption which was imposed on functions from set $F(t)$ is the necessity of settling these functions during the transient state of the original time-invariant filter. This condition can be written as follows:

$$\forall_{t>t_{u\alpha}}\ F(t)=\overline{F}\pm\alpha \tag{16}$$

where: $t_{u\alpha}$ - settling time of a constant parameter filter with an accuracy of α. Relation (16) is responsible for keeping frequency assumptions of designed time-varying filter.

The introduction of time-varying parameters requires examination of the stability of the systems with element containing varying parameters. Paper [6] present the proof of a theorem saying that if for the 2-nd order system the functions $\omega_{0i}(t)$ and $\beta(t)$ have the same sign and:

$$\lim_{t\to\infty}\frac{d\omega_{0i}(t)}{dt}\to 0 \tag{17}$$

then stability can be determined in the same manner as for time-invariant systems. It allows us to claim that if the filter structure contains elements with varying parameters, then for the time going to infinity the values of varying parameters converge to limit values following from the Butterworth approximation and the filter stability can be examined as in the time-invariant case. Since the Butterworth approximation guarantees stability of time-invariant filter one can skip the stability issue for the parametric filter.

In order to shorten the filter transient state one assumes functions from set $F(t)$ as follows:

$$F(t)=d\cdot\overline{F}\cdot\left[1-\frac{d-1}{d}\cdot h(t)\right] \tag{18}$$

where: F - value of parameter from set F following from the Butterworth approximation, d - variation range of the functions from set $F(t)$ described by the relation:

$$d=\frac{F(0)}{\overline{F}=F(\infty)} \tag{19}$$

Function $h(t)$ describes the step respond of the second order system:

$$h(t)=L^{-1}\left[\frac{1}{s}\cdot\frac{1}{\dfrac{1}{\omega_{0f}{}^2}s^2+\dfrac{2\beta_f}{\omega_{0f}}s+1}\right] \tag{20}$$

where: β_f is the parameter which determines oscillations of functions from set $F(t)$, ω_{0f} is the parameter which determines the range of the speed of above-mentioned functions, and L^{-1} is the inverse Laplace transform.

The best results in shortening the transient state of phase-compensated Butterworth filters were obtained while parameters ω_{0i} and $1/T$ were varied according to the same function $F(t)$. Fig. 3 presents responses to the noised rectangular input signal for the original Butterworth filter and the phase-compensated time-varying Butterworth filter.

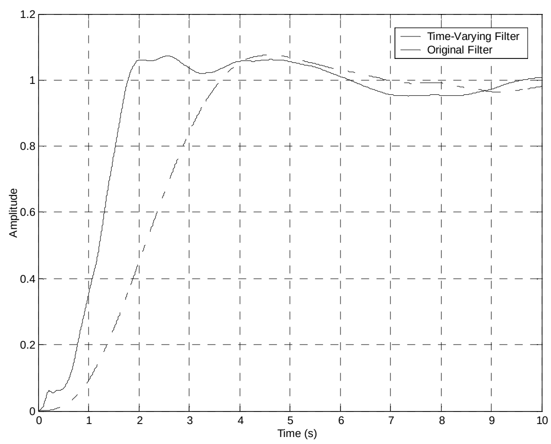

Fig. 3. Filter responses to the noised rectangular input signal.

4. CONCLUSION

As it has been proven, application of time-varying coefficients in the low-pass phase-compensated Butterworth filters causes considerable shortening of the settling time. The best results of shortening of the settling time were obtained by varying in time the characteristic frequency ω_{0i} and the inverse of the time constant T according to the same function. It seems that further examinations of time-varying Butterworth filters are needed. Especially, the problems of optimal selection of variation range and the speed range of functions are open. Nevertheless already this paper proves possibilities and practical usefulness of the proposed filter concept as a signal processing instrument.

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