A SYSTEMATIC APPROACH TO ESTIMATING UNCERTAINTY IN PRESSURE MEASUREMENT

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Abstract: For any measurement to be meaningful, the result of the measurement must be accompanied with a statement of its uncertainty. The evaluation of uncertainties associated with pressure measurement is sometimes complex but an important task. This paper presents a systematic approach for estimating measurement uncertainty by providing a worked example for the case of pressure measurement by a pneumatic dead-weight tester.

Keywords: Pressure, uncertainty, influence parameters.

1. INTRODUCTION

The *Uncertainty* of a measurement is defined as a "parameter, associated with the result of a measurement that characterizes the dispersion of values that could reasonably be attributed to the measurand". As such uncertainty is a measure of possible error in the estimated value of the measurement result and evaluation of uncertainty should, therefore, be an integral part of the measurement process. For a measurement to be considered traceable, it is important that the measurement results be accompanied with a statement of uncertainty.

Depending on the pressure-measuring instrument, the estimation of measurement uncertainty could often be complex because of large number of parameters that can influence the measurement uncertainty. This paper provides an example of a typical evaluation of measurement uncertainty for the case of pressure measurement by a dead-weight tester, which is the most common device used for generating very accurate pressures. The uncertainty evaluation presented here follows generally accepted criteria as expressed in the *ISO Guide to the Expression of Uncertainty in Measurement (GUM)*.

2. UNCERTAINTY PROPAGATION

The measurement uncertainty depends on the uncertainty of the input variable parameters and their functional relationship to the measurand. The transfer of uncertainties from the input variables to the output through the functional relationship is called the uncertainty propagation. The quantum of propagation depends on the uncertainties associated with the input variables, their statistical distributions, their correlation and the nature of the functional relationship.

2.1. Model equation for measurement

The functional relationship between a measurand Y and the input parameters X_i *(i = 1,2, …n)* can be represented as:

$$Y = f\left(X_1, X_2, ...X_n\right) \tag{1}$$

Where the function f will depends upon the particular measurement procedure.

Input parameters could be quantities whose estimates and associated uncertainties are either evaluated during the current measurement process or imported from external sources such as a calibration certificate.

2.2. Uncertainty Evaluation Of Influence Parameters

The *ISO Guide to the Expression of Uncertainty in Measurement* classifies uncertainty evaluations as either Type A or Type B, based on method of evaluation. Type A evaluations are based on statistical evaluation of data collected from repeated measurements while Type B evaluations are based on available information from sources such as calibration certificates, system specifications etc.

2.2.1. Type A Uncertainty Evaluation

In most cases, the arithmetic mean or average of n independent observations of a quantity q is an estimation of its true value. The mean of n measurements is given by:

$$\overline{q} = \frac{1}{n}\sum_{k=1}^{n} q_k \tag{2}$$

The standard deviation of the mean is given by:

$$u(\overline{q}) \equiv \sigma_{\overline{q}} = \sqrt{\frac{\sum_{k=1}^{n}\left(\overline{q} - q_k\right)^2}{n(n-1)}} \tag{3}$$

The standard deviation of the mean quantifies how well $\overline{q}$ estimates the expected value of q and is, therefore, used as a measure of the uncertainty of $\overline{q}$. Thus, for an input quantity X_i **,** estimated from n independent measurements $X_{i,k}$, the standard uncertainty $u(x_i)$ of its estimate $x_i = \overline{X}_i$ is $u(x_i) = \sigma_{\overline{X}_i}$, and is calculated according to above equation.

2.2.2. Type B Uncertainty Evaluation

The standard uncertainty $u(x_i)$ for an estimate x_i of an input quantity X_i that has not been obtained from repeated observations, is evaluated by other methods based on the available data about the variability of X_i . The proper use of available information for Type B evaluations requires insight based on experience and the scientific judgment of the metrologist.

In pressure measurements, most uncertainty evaluations are type B. The following two cases could be of particular interest:

i) If the estimate x_i is taken from a calibration certificate or manufacturer's specification and its uncertainty is quoted as some multiple of a standard deviation, than the standard uncertainty $u(x_i)$ is simply the quoted value divided by the multiplier.

ii) If it is possible to estimate only the upper and lower limits a_+ and a_- of an input parameter X_i, then it is reasonable to assume a rectangular distribution. Then the estimated value and its standard uncertainty are given by:

$$x_i = \frac{1}{2}\left(a_+ + a_-\right);$$

$$\text{and, } u(x_i) = \sqrt{\frac{\left(a_+ - a_-\right)^2}{12}} \qquad 4)$$

For example, a reading from a digital read-out of resolution a has a standard uncertainty of $a/\sqrt{12}$; the digital reading is considered to be a rectangular distribution bounded by $a_+=+a/2$ and $a_-= - a/2$.

2.3. *Combined standard uncertainty*

For a simple case of uncorrelated or independent input quantities, the combined standard uncertainty $u_c(y)$ of the output estimate y is given by

$$u_c^2(y) = \sum_{i=1}^{N}\left[\frac{\partial f}{\partial x_i}\right]^2 u^2(x_i) = \sum_{i=1}^{N} c_i^2 u^2(x_i) \qquad (5)$$

where $u(x_i)$ are the standard uncertainties of the input quantities x_i, and $c_i = \partial f/\partial x_i$, often called sensitivity coefficients, are the partial derivatives of the model equation.

2.4. *Expanded uncertainty*

The expanded uncertainty U is defined as the combined standard uncertainty multiplied by a coverage factor k:

$$U = k\, u_c(y) \qquad (6)$$

The value of k is chosen depending upon the required confidence level. In cases where normal (Gaussian) distribution can be attributed to the measurand, the coverage factor $k = 2$ corresponds to a confidence level of approximately 95%.

3. WORKED EXAMPLE: DEAD WEIGHT TESTER

Dead-weight testers (DWT) also known as pressure balances, are most common devices used for calibrating a wide range of pressure transducers. The device measures pressure directly in terms of fundamental units of force and area. In principle, the pressure is measured under conditions of equilibrium between the force produced by a mass m and the resulting force produced by the pressure exerted at base of a cylindrical piston of effective area A_e.

3.1. *Pressure measurement equation*

The pressure generated by a DWT is given as:

$$P = \frac{\left[\Sigma m_i(1-\rho_a/\rho_{m_i})\right]g_l + \gamma C}{A_o\left(1+\beta P_i\right)[1+(\alpha_p+\alpha_c)(t-t_{ref})]} + g_l h\left(\rho_f - \rho_a\right) \qquad (7)$$

where;

m_i is the mass of the i_{th} weight loaded on the Piston,
ρ_{m_i} is the corresponding density of the mass material,
ρ_a is the density of ambient air,
ρ_f is the density of working fluid at the measured pressure
g_l is local acceleration due to gravity,
γ is the surface tension of the working fluid,
C is the circumference of the piston,
A_o is the effective area of the piston-cylinder assembly usually quoted in a calibration certificate, at zero pressure and some reference temperature t_{ref},
β is the pressure distortion coefficient,
P_i is the nominal measured pressure,
α_p and α_c are the thermal expansion coefficients of Piston and Cylinder materials respectively,
t is the temperature of the piston-cylinder assembly measured during the operation, and
h is the height between the piston base and selected reference level.

For a pneumatic dead weight tester the effects of surface tension and pressure distortion are negligible and if all the mass plates are considered to be made of same material, equation 6 reduces to:

$$P = \frac{m\, g_l(1-\rho_a/\rho_m)}{A_o[1+(\alpha_p+\alpha_c)(t-t_{ref})]} + g_l h\left(\rho_f - \rho_a\right) \qquad (8)$$

where, $m = \sum_i m_i$ is the total mass on the piston and ρ_m is the density of mass material.

3.2. *Uncertainty evaluations*

Adding the uncertainty contributions due to various influence quantities in the above equation, the combined standard uncertainty in pressure is given by:

$$u_c^2(P)=c_m^2u^2(m)+c_{g_l}^2u^2(g_l)+c_{\rho_a}^2u^2(\rho_a)+c_{\rho_m}^2u^2(\rho_m)+c_{A_0}^2u^2(A_0)+$$

$$c_{\alpha_p}^2u^2(\alpha_p)+c_{\alpha_c}^2u^2(\alpha_c)+c_t^2u^2(t)+c_h^2u^2(h)+c_{\rho_f}^2u^2(\rho_f) \quad (9)$$

$u(m)$, $u(g_l)$ etc. are the standard uncertainties of the input quantities, and c_m , etc. are the sensitivity coefficients.

The sensitivity coefficients are obtained by partially differentiating the measurement equation, and are given as:

$$c_m = \frac{\partial p}{\partial m},\ c_{\rho_m} = \frac{\partial p}{\partial \rho_m},\ c_{g_l} = \frac{\partial p}{\partial g_l},\ \text{etc.}$$

3.2.1. Uncertainty of influence quantities

i) Mass of loading plates – *m*

The uncertainty and the mass value for each loading weight are usually obtained from a calibration certificate.

For uncorrelated value of masses, the combined mass uncertainty of a stack of loading weights is obtained by combining the mass uncertainty of each plate and piston in quadrature (root-sum-square). However, if all the loading plates for the dead weight tester are calibrated in the same calibration lab, which is usually the case, the mass values can be considered totally correlated. The combined mass uncertainty of a stack is then obtained by a linear sum of individual uncertainties.

$$u_c(m)=\sum_i u(m_i) \quad (10)$$

For example, if the (relative) standard uncertainty of each plate in a stack is specified as 6 ppm, then the (relative) combined mass uncertainty of the stack will also be 6 ppm.

ii) Acceleration due to gravity - g_l

The local acceleration due to gravity g_l at the test site may be computed from the following formula:

$$g_l = g_0\left(1+b_1\sin^2\phi - b_2\sin^2 2\phi\right)-3.086\times10^{-6}h \quad (11)$$

where:

g_0 = 9.780 318 4 m/s^2 is the gravity value at the equator,

b_1 = 5.302 4×10^{-3} and b_2 = 5.9×10^{-6} are constants

ϕ and h are the latitude and height, in metres, above sea level of the local site.

The uncertainty in g_l can then be estimated by applying equation 2, taking into account the uncertainty of influence quantities in above equation. Uncertainty in g_l thus calculated could be of the order of 100 ppm. For high accuracy work the gravity value should be determined experimentally at the test site.

iii) Air density –ρ_a

The density of air varies with the temperature, pressure and humidity, and can be computed from the following simplified equation (adopted from reference 4):

$$\rho_a = \frac{0.0034848\,p \ -0.009024\,R_h\exp(0.0612\,t)}{t+273.15} \quad (12)$$

where, p_{atm} is the atmospheric pressure in Pa
t is the temperature of the ambient air in °C
R_h is the percent relative humidity, and

The combined standard uncertainty of the air density is calculated according to following equation:

$$u(\rho_a)=\rho_a\left[\left(\frac{u_f}{\rho_a}\right)^2+\left(\frac{u_p}{p}\right)^2+\left(\frac{u_t}{T}\right)^2+\left(\frac{u_{Rh}}{10^4}\right)^2\right]^{1/2} \quad (13)$$

where, u_f/ρ_a = 2×10^{-4} is the relative uncertainty of the approximate formula, and u_p , u_t and u_{Rh} are the standard uncertainties of pressure, temperature and relative humidity measurement in question, and T=t+273.15 is the absolute temperature of the ambient air.

For example, for following measurement values and standard uncertainties:

pressure: p = 100727 Pa ; u_p = 50 Pa
temperature: t = 22.0°C (T=295.15); u_t = 0.5 °C
relative humidity: R_h = 31.2 % ; u_{Rh} = 2 %

The air density and standard uncertainty calculated from equation (12) and (13) are:

ρ_a = 1.1854 kg/m^3 and $u(\rho_a)$ = 0.0022 kg/m^3

iv) Density of mass material – ρ_m

The standard uncertainty in the density of the material can be taken from manufacturers specification, with a rectangular distribution as a conservative bound. The standard uncertainty $u(\rho_m)$ for stainless steel plates whose density is stated as (7800 ± 200) kg/m^3, is given by:

$$u(\rho_m) = 200/\sqrt{3} = 115.5\ \text{kg/m}^3$$

v) Effective Area – A_0

The effective area of the piston-cylinder assembly is usually obtained from the calibration certificate, either supplied by the manufacturer or a standards laboratory, with a stated expanded uncertainty statement. The standard uncertainty in the effective area is then:

$$u(A_0)=\frac{Expanded\ Uncert.}{Coverage\ factor}$$

For the present example, A_0 = 335.920×10^{-6} m^2 and $u(A_0)$ = 5.0×10^{-9} m^2

vi) Expansion coefficient – α_p and α_c

The standard uncertainty in the thermal coefficient of piston $u(\alpha_p)$and cylinder $u(\alpha_c)$ can be taken to be 10% of the manufacturers specification, with a rectangular distribution as a conservative bound. For this example where both the piston and cylinder are made of tungsten carbide:

$\alpha_p = \alpha_c = 11.2\times10^{-6}$/K, and $u(\alpha_p)$ and $u(\alpha_c)$ can be estimated as:

$$u(\alpha_p) = u(\alpha_c) = \frac{10\% \cdot (11.2\times10^{-6})}{\sqrt{3}} = 6.47\times10^{-7}/\text{K}$$

vii) Temperature of the piston-cylinder assembly - t

The combined standard uncertainty attributed to temperature measurement of the piston-cylinder assembly should include uncertainty components due to calibration of the sensor, and resolution and drift of the read out device, and is given as:

$$u_c^2(t) = u^2(t_{cal}) + u^2(t_{read}) + u^2(t_{drift}) \quad (14)$$

For example, if a temperature sensor with a calibration uncertainty of 10 mK is used with a digital readout that has a resolution of 10 mK and its drift is specified as 25 mK, the various uncertainty components can be evaluated as follows:

- Calibration of the sensor – the calibration uncertainty of the temperature sensor is 10 mK, giving
  $$u(t_{cal}) = 0.010\text{K}$$
- Reading resolution – the resolution of the digital device is 10 mK, giving
  $$u(t_{readl}) = \frac{0.01}{\sqrt{12}} = 0.003\text{ K}$$
- Drift– the drift between calibrations of the measuring instrument is 25 mK. Assuming rectangular distribution, therefore:
  $$u(t_{drift}) = \frac{0.025}{\sqrt{3}} = 0.015\text{ K}$$

The combined standard uncertainty attributed to temperature measurement is then given as:

$$u_c(t) = \sqrt{(0.010)^2 + (0.003)^2 + (0.015)^2} = 0.018\text{ K}$$

viii) Density of the working fluid – ρ_f

The density of the fluid at the working pressure can be calculated using the following equation of state for ideal gases:

$$\rho_f = \frac{pM}{RT} \quad (15)$$

where, M is molar mass of the gas (28.0135 x 10^{-3} kg.mol^{-1} for nitrogen), R is the gas constant (8.314 51 J mol^{-1}K^{-1}), and T is the absolute temperature in Kelvin. The expected uncertainty could be assumed to be 10% of the density calculated from the above equation. For example, for dry nitrogen at 22.3 ^{o}C and 100 kPa, the calculated density is 1.1404 kg/m^3, and

$$u(\rho_f) = \frac{0.1140}{\sqrt{3}} = 0.06\text{ kg/m}^3$$

3.2.2. *Sensitivity Coefficients*

The sensitivity coefficients are evaluated by performing partial derivatives of equation (8).

$$\frac{\partial P}{\partial m} = \frac{P - h \cdot g_l(\rho_f - \rho_a)}{m}, \quad \frac{\partial P}{\partial g_l} = \frac{P}{g_l}$$

$$\frac{\partial P}{\partial \rho_a} = \frac{[-P + h.g_l(\rho_m - \rho_f)]}{\rho_m - \rho_a}, \text{ etc.}$$

3.2.3. *Combined standard uncertainty*

The standard uncertainty and the sensitivity coefficients of various influence quantities are tabulated in the following table.

Input variables and their estimates	Standard Uncertainty	Uncertainty contribution of X_i
Variable (X_i)	$u(x_i)$	$\lvert u(x_i).\partial p/\partial(x_i)\rvert$
m – 3.426 3 (kg)	1.94 x10^{-5}	0.57
g_l – 9.80616 (m/s^2)	4.9 x10^{-6}	0.05
ρ_a – 1.1854 (kg/m^3)	2.20x10^{-3}	0.05
ρ_m – 7800 (kg/m^3)	115.5	0.22
α_p – 11.2×10^{-6} (/K)	6.47×10^{-7}	0.20
α_c – 11.2×10^{-6} (/K)	6.47×10^{-7}	0.20
t–296 (K)	1.8 ×10^{-2}	0.04
A_0 – 335.920 (mm^2)	5.0 ×10^{-3}	1.50
ρ_f – 1.1404 (kg/m^3)	0.06	0.60
h – 1 (m)	5.0	0.002

The combined standard uncertainty $u_c(P)$ in pressure can now be computed from equation (9) and is given as:

$$u_c(P) = \sqrt{(0.57)^2 + (0.05)^2 + (0.05)^2 + ...} = 1.8\text{ Pa}$$

3.2.4. *Expanded uncertainty*

The final uncertainty to be reported with this measurement is the expanded uncertainty U obtained by multiplying the combined standard uncertainty $u_c(P)$ by a suitable coverage factor k. In most cases $k = 2$ provides a level of confidence approximately 95%. For this example the expanded uncertainty in the measured pressure is:

$$U = k\, u_c(P) = 2 \times 1.8 = 3.6 \text{ Pa}$$

and, the measured pressure is reported as:

$$P = 99.999\,6 \text{ kPa} \pm 3.6 \text{ Pa}$$

4. CONCLUSION

The expanded uncertainty of pressure measurement by a pneumatic dead-weight tester has been evaluated. A step-by-step procedure for evaluation of the measurement uncertainty has been presented. The procedure can be readily adopted for more complex cases such as in hydraulic dead weight testers or in cross-float experiments.

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