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# MATHEMATICAL MODELLING OF SAMPLING FLOWMETERS

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*Abstract: Sampling flowmeters (called sometimes insertion flowmeters) are devices for measuring of flow-rate. The volume flow-rate is inferred on the base of the result of velocity measurement in certain point in the flow area. The two models of velocity distribution are introduced and for four possibilities of sensor placing in the pipe the formulas for distances of velocity sensor placing in the pipe are derived. The volume flow-rate is the product of measured local velocity, the flow area and the sensitivity factor. The formulae for sensitivity factor are derived for these four situations, and the volumetric flow-rate can be calculated. In conclusion are introduced some problems connected with volume flow-rate measurement with sampling flowmeter.*
*Keywords: sampling flowmeter, insertion flowmeter, non-intrusive flowmeter*

## 1 INTRODUCTION

The measurement of flow-rate of fluids in pipes is getting more difficult because pipe diameters are growing up and thus the costs of in-line flowmeters (full bore flowmeters) also are growing. It concerns such flowmeters as orifice flowmeters, turbine and electromagnetic ones. These flowmeters are installed in the line. The examples of sampling flowmeters are: Pitot static probe, turbine probe, averaging impact tube, thermal resistance flowmeter, ultrasonic flowmeter [1, 2]. From metrological point of view the accuracy and reliability are the essential criteria, and the working principle is not so important. The accuracy of sampling flowmeter is worst than accuracy of in-line flowmeters, but reliability of sampling flowmeters is rather greater than the reliability of in-line flowmeters.

The cost of sampling flowmeter is practically independent on pipe diameter, but the cost of in-line flowmeter is almost proportional to the pipe diameter. For the in-line flowmeters the output signal depend on the whole flow-rate in the cross section of the pipe. For sampling flowmeters the flow-rate is inferred from the velocity measured at the selected point or place in the flow stream [1].

The sampling flowmeters can be divided for non-intrusive flowmeters [3] and insertion flowmeters [2, 3, 4]. The possibilities of applications are very important and in [2] are presented four categories of flowmeters: 1) flowmeters with wetted moving parts, 2) flowmeters with wetted no moving parts, 3) obstructionless flowmeters, 4) flowmeters with sensors mounted external to the pipe. The flowmeters of the first category are in-line flowmeters (full-bore flowmeters). Second category includes in-line flowmeters (differential pressure as well as insertion flowmeters (target, thermal). The third category includes in-line flowmeters (Coriolis mass and magnetic) as well as sampling, non-intrusive flowmeters (ultrasonic) with wetted sensors. To the fourth category belongs for example ultrasonic clamp-on flowmeters (Doppler and time of flight). Spitzer [1] gives also another division: for four types: 1) volumetric, 2) velocity, 3) inferential, 4) mass. For velocity type the total flow-rate is determined by the multiplication of the velocity measurement of the velocity measured in the flowmeter primary device by the area through which the fluid flows. In [5] the classification of flowmeter primary devices of sampling flowmeters is introduced. The volume flow-rate can be calculated from: 1) the point velocity measurement (one or more), 2) the average velocity measurement over some surface in the point velocity measurement (one or more), 3) the average velocity measurement over some surface in the flow area, 4) the average velocity measurement along a segment (or some segments) in the flow area. In [6] is introduced a mathematical model of an ultrasonic flowmeter primary device with the measurement of an average velocity along a segment in the flow area. In [1] is introduced a mathematical model of a point velocity measurement (useful for example for turbine insertion flowmeter). The flow-rate calculation is possible when we know the relationship between the measured velocity and the total volume flow-rate. The accuracy of calculation of the flow-rate depends on the accuracy of the mathematical model of velocity distribution. The accuracy of this model it is the first and essential problem, but the accuracy of flow-rate estimation depends also on the place in which the velocity is measured and of course depend on velocity sensor accuracy. The work concerns some theoretical problems of using of sampling flowmeters.

## 2 MATHEMATICAL MODEL OF VELOCITY DISTRIBUTION

For a turbulent flow, the Prandtl formula (power-low velocity profile) is most frequently used:

$$v = v_o (1 - r/R)^{1/n}, \tag{1}$$

where: $R$ - pipe radius, $r$ - current radius, $v_o$ - velocity along the pipe axis, $n$ - number which depends on Reynolds number and roughness of the pipe wall.
Miller [5] proposes that the value of $n$ for smooth pipes can be calculated as a function of the Reynolds number:

$$n = 1{,}66 \log \mathrm{Re}. \tag{2}$$

In [1,2] are given the following formulae:

$$n = 3{,}299 + 0{,}3257 \ln \mathrm{Re} \qquad \text{for Re} < 400\,000, \tag{3}$$

$$n = 5{,}5365 + 5{,}498 \times 10^{-6} (\ln \mathrm{Re})^5 \qquad \text{for Re} > 400\,000. \tag{4}$$

The analysis of Prandtl formula shows [8], that it estimates properly velocity profile near the pipe wall, but is not adequate near the pipe axis. The formula proposed in [9, 10] is adequate for laminar flow ($m$ = 2) as well as for turbulent flow:

$$v = v_o [1 - (r/R)]^m. \tag{5}$$

The authors analysis of many velocity profiles measured in various conditions give approximate expression:

$$m = 0{,}75n + 0{,}5. \tag{6}$$

## 3 POSSIBLE PLACES FOR VELOCITY MEASUREMENT IN THE PIPE CROSS-SECTION

For an ideal fluid and steady flow the velocity profile is constant throughout the cross-section of the pipe [2]. The velocity sensor can be placed in any point of cross-section and measured velocity will be equal to average velocity. The measured value, the volume flow-rate is product of measured velocity and the area of cross-section.

In real applications the velocity profile is not uniform and the proper place for velocity measurement must be chosen. In [1,2] are introduced two possibilities: critically positioned application and centreline positioned application. Author proposed [11] three possibilities more: equal area position, equal flow-rate position and optimum position.

### 3.1 Centreline position

The centreline position seems to be natural position. The probe sensor is placed at the centre of the pipe and the centreline velocity is measured [1]. The greatest difference between measured velocity and average velocity is two times greater than average velocity (for laminar flow). When the Reynolds number grows for turbulent flow the measured velocity approaches to average velocity.

### 3.2 Critical position

The critical position means placing of velocity sensor in the point where local velocity is equal to average velocity. For velocity distribution described by Prandtl formula (1) the average velocity can be calculated using the formula:

$$v_A = v_o \, 2n^2 \big/ \left[(n+1)(2n+1)\right]. \tag{7}$$

Critical position is calculated from (1) and (7):

$$r_p / R = 1 - \left\{2n^2 \big/ \left[(n+1)(2n+1)\right]\right\}^n. \tag{8}$$

For velocity distribution described by universal formula (5) the average velocity can be calculated from:

$$v_A = v_o m / (m+2). \tag{9}$$

For velocity distribution described by formula (5) the critical position can be calculated from:

$$r_p / R = \sqrt[m]{2/(m+2)}. \tag{10}$$

In the table 1 are presented results of calculation according to formula (8) and (10), but by an assumption that the relationship between $m$ and $n$ is expressed by formula (6). The data in the table 1 show, that the assumed shape of the mathematical model of velocity distribution will decide about the value of critical position $r_p/R$. The change of $r_p/R$ for Prandtl formula (1) with Reynolds number are smaller than for model expressed by formula (5). For turbulent flow and the ratio of Reynolds number

1000:1 (n=11 and n=6) the difference in critical position is about 1,3%. In [1, 2] is not analysed the influence of roughness of the pipe wall. This influence can be calculated on the base of the velocity distribution shape factor *K*. The influence of the Reynolds number and roughness of the pipe wall on the factor *K* are presented in [5, 8, 12]. The values of number *n* are calculated for Reynolds number Re = $5.10^4$ and the changes of relative roughness *k*/*R* from 0 to 0,01. Calculation of $r_p/R$ on the base of formula (8) gives: $r_p/R$ = 0,7858 and $r_p/R$ = 0,7646 what gives the changes of critical position + 1,2%.

Table 1. Critical position for velocity distribution described by formulae (1) and (5).

| *n* | Re(*n*) | $[r_p/R](n)$ | *m*(*n*) | $[r_p/R](m)$ |
|---|---|---|---|---|
| - | - | - | 2 | 0,7071 |
| - | - | - | 3 | 0,7368 |
| - | - | - | 4 | 0,7598 |
| 6 | $4\text{x}10^3$ | 0,7547 | 5 | 0,7783 |
| 7 | $16\text{x}10^3$ | 0,7577 | 5,75 | 0,7901 |
| 8 | $66\text{x}10^3$ | 0,7600 | 6,5 | 0,8004 |
| 9 | $264\text{x}10^3$ | 0,7618 | 7,25 | 0,8096 |
| 10 | $1\ 057\text{x}10^3$ | 0,7633 | 8 | 0,8178 |
| 11 | $4\ 232\text{x}10^3$ | 0,7645 | 8,75 | 0,8251 |
| 12 | $16\ 940\text{x}10^3$ | 0,7655 | 9,5 | 0,8318 |

## 3.3 Equal area position

This art of one point velocity measurement depends on that the velocity sensor is placed on the circle, which divides flow area for two parts and the distance of this point from the circle centre is:

$$r_{\mathrm{A}} / R = 1/\sqrt{2}. \tag{11}$$

This art of placing is proper for almost piston velocity distribution, which can be slightly distorted. In such situation there is no sense to find the critical position. This is used in velocity area method of volume flow-rate calculation on the base of multipoint velocity measurements.

## 3.4 Equal flow-rate position

The velocity sensor is placed on the circle, which divides the flow area for two parts in which volume flow-rates are equal:

$$q_{v\mathrm{q}} = \int_0^{r_\mathrm{q}} v(r) 2pr\mathrm{d}r = q_v / 2. \tag{12}$$

For Prandtl velocity distribution model (1) the distance $r_q/R$ can be calculated from:

$$(1 - r_\mathrm{q} / R)^{(1+2n)/n} + [(1+2n)/n](r_\mathrm{q} / R)(1 - r_\mathrm{q} / R)^{(1+n)/n} = 0{,}5. \tag{13}$$

For universal model of velocity distribution (5):

$$[(m+2)/m](r_\mathrm{q} / R)^2 - (2/m)(r_\mathrm{q} / R)^{m+2} = 0{,}5. \tag{14}$$

The equal flow-rate positions were calculated on the base of (13) and (14) with iterative method and the results of calculation are presented in the table 2.

Table 2. Equal area positions for velocity distribution described by formulae (1) and (5).

| *n* | Re(*n*) | $[r_q/R](n)$ | *m*(*n*) | $[r_q/R](m)$ |
|---|---|---|---|---|
| - | - | - | 2 | 0,5412 |
| - | - | - | 3 | 0,5691 |
| - | - | - | 4 | 0,5893 |
| 6 | $4\text{x}10^3$ | 0,6620 | 5 | 0,6046 |
| 7 | $16\text{x}10^3$ | 0,6681 | 5,75 | 0,6139 |
| 8 | $66\text{x}10^3$ | 0,6728 | 6,5 | 0,6217 |
| 9 | $264\text{x}10^3$ | 0,6764 | 7,25 | 0,6283 |
| 10 | $1\ 057\text{x}10^3$ | 0,6794 | 8 | 0,6341 |
| 11 | $4\ 232\text{x}10^3$ | 0,6818 | 8,75 | 0,6391 |
| 12 | $16\ 940\text{x}10^3$ | 0,6839 | 9,5 | 0,6435 |

## 3.5 Optimum position

Very often we now, that the volume flow-rate practically changes from $q_{v\min}$ to $q_{v\max}$ or that the volume flow-rate changes insignificantly about nominal value. Optimum position it is such position, which gives minimal measurement error. For nominal flow-rate the velocity distribution can change also because of roughness of the pipe wall changes. In this situation we can estimate the scope of roughness changes and use the velocity distribution model in which these changes are taken into account. In [8] author proposed 5 criteria for error definition and we can choose what kind of error will be minimised (for example maximum error, the integral of error, average error...).

# 4 CALCULATION OF VOLUME FLOW-RATE ON BASE OF RESULT OF THE POINT VELOCITY MEASUREMENT

## 4.1 Introduction

Volume flow-rate in sampling method of flow measurement is calculated from:

$$q_v = v_{\mathrm{i}} k_{\mathrm{i}} A, \tag{15}$$

where: $v_{\mathrm{i}}$ - the velocity value measured with the "i" art of point velocity measurement, $k_{\mathrm{i}}$ - sensitivity factor for "i" art of point velocity measurement, $A$ - cross-section area of the stream in the pipe.
The sensitivity factor is defined in the following way:

$$k_{\mathrm{i}} = v_A / v_{\mathrm{i}}, \tag{16}$$

where: $v_A$ - average velocity in cross-section of the stream in the pipe.
For volume flow-rate calculation can be assumed constant value of the sensitivity factor (for example for nominal value of volumetric flow) or this value can be a function of measurement value $q_v$ or of the actual Reynolds number.

## 4.2 Centreline position

The average velocity $v_A$ is calculated from formula (7), and the velocity $v_{\mathrm{i}}$ is the velocity in the pipe axis $v_{\mathrm{o}}$. For Prandtl formula (1) the sensitivity factor is:

$$k_{\mathrm{o}} = 2n^2 \big/ \left[(n+1)(2n+1)\right]. \tag{17}$$

For universal equation of velocity distribution (5) the average velocity can be calculated from (9) and the sensitivity factor is:

$$k_{\mathrm{o}} = m/(m+2). \tag{18}$$

The volume flow-rate is calculated from (15) and (17) or (18).

## 4.3 Critical position

For this position the distance of the sensor is calculated from (8) and the position of the sensor must change when the volume flow-rate changes. The practical situation will be different: we assume some volume flow-rate as nominal and for this value we estimate the sensor position, for which the value of sensitivity factor $k_{\mathrm{p}} = 1$. The measured velocity:

$$v_{\mathrm{p}} = v_{\mathrm{o}} (1 - r_{\mathrm{p}} / R)^{1/n}. \tag{19}$$

The nominal position on the base of (8) will be as follows:

$$r_{\mathrm{pn}} / R = 1 - \left\{2n_{\mathrm{n}}{}^2 \big/ \left[(n_{\mathrm{n}}+1)(2n_{\mathrm{n}}+1)\right]\right\}^{n_{\mathrm{n}}}. \tag{20}$$

From (19) and (20) the measured velocity can be calculated from:

$$v_{\mathrm{pn}} = v_{\mathrm{o}} \left\{2n_{\mathrm{n}}{}^2 \big/ \left[(n_{\mathrm{n}}+1)(2n_{\mathrm{n}}+1)\right]\right\}^{n_{\mathrm{n}}/n}. \tag{21}$$

The sensitivity factor can be calculated from (7) and (21):

$$k_{\mathrm{pn}} = \left\{2n^2 \big/ \left[(n+1)(2n+1)\right]\right\} \left\{\left[(n_{\mathrm{n}}+1)(2n_{\mathrm{n}}+1)\right] \big/ \left[2n_{\mathrm{n}}{}^2\right]\right\}^{n_{\mathrm{n}}/n}. \tag{22}$$

This factor depends on actual value of $n$, which can be calculated with iterative method. In the first approximation we assume nominal value of $n_{\mathrm{n}}$ and we calculate the volume flow-rate for $k_{\mathrm{p}} = 1$. The value of volume flow-rate in this way is the base for Reynolds number calculation. For Reynolds number the kinematic viscosity must be known. The viscosity depends on the temperature and composition of the fluid. On the base of calculated value of Re and from (2) or (3) and (4) the value of $n$ is calculated and from (22) we receive second approximation. Formulae (2), (3) or (4) are for smooth pipes. When the pipe wall can not be assumed as smooth, it is possible to use the diagram from [8] or [12] and on the base of value of velocity distribution shape coefficient to calculate the appropriate value of *n*. For the velocity distribution expressed by (5) the velocity measured with help of the sensor will be:

$$v_{pn} = v_o \left\{1 - \left[2/(m_n + 2)\right]^{m/m_n}\right\}. \quad (23)$$

From formulae (9) (16) and (23) the sensitivity factor is estimated:

$$k_{pn} = m \Big/ \left\{(m+2)\left[1 - \left[2/(m_n + 2\right]^{m/m_n}\right]\right\}. \quad (24)$$

Practical problems connected with using of this factor are similar to problems connected with using of sensitivity factor described with (22).

## 4.4 Equal area position

The velocity is measured in the distance from pipe axis expressed with formula (11), the average velocity for Prandtl model is expressed with formula (7), and the sensitivity factor is:

$$k_A = 2n^2 \Big/ \left[(n+1)(2n+1)\left(1 - 1/\sqrt{2}\right)^{1/n}\right]. \quad (25)$$

For universal model for velocity distribution (5) the velocity measured with sensor:

$$v_A = v_o \left[1 - \left(1/\sqrt{2}\right)^m\right]. \quad (26)$$

Putting (9) and (26) to (16) we receive the sensitivity factor:

$$k_A = m \Big/ \left\{(m+2)\left[1 - \left(1/\sqrt{2}\right)^m\right]\right\}. \quad (27)$$

## 4.6 Equal flow-rate position

For Prandtl velocity distribution model (1) the sensitivity factor can be calculated on the base of formulae (1), (7) and (16):

$$k_q = 2n^2 \Big/ \left[(n+1)(2n+1)(1 - r_q/R)^{1/n}\right], \quad (28)$$

where the value $r_q/R$ is calculated from formula (13).
For universal velocity distribution model (5) the sensitivity factor can be calculated on the base of formulae (1), (7) and (16):

$$k_A = m \Big/ (m+2)\left[1 - (r_q/R)^m\right], \quad (29)$$

where the value $r_q/R$ is calculated from formula (14).
In the case of equal flow-rate position appear the same problems like for the critical position.

## 4.6 Optimum position

For Prandtl velocity distribution model (1) the sensitivity factor can be calculated on the base of formulae (1), (7) and (16):

$$k_{opt} = 2n^2 \Big/ \left[(n+1)(2n+1)(1 - r_{opt}/R)^m\right]. \quad (30)$$

For the universal velocity distribution model (5) the sensitivity factor can be calculated on the base of formulae (5), (9) and (16):

$$k_{opt} = m \Big/ \left\{(m+2)\left[1 - (r_{opt}/R)^m\right]\right\}. \quad (31)$$

Optimum position $r_{opt}/R$ is calculated for taken into account error criterion and for example for Prandtl velocity distribution model (1) in [11] the minimum of absolute value was taken into account and the relative method error connected with sensor placing was calculated.

## 5 CONCLUSIONS

- Two mathematical models (Prandtl and universal; which is suitable for laminar flow and turbulent flow) estimating the velocity distribution in a pipe are introduced. The presented function between parameters characterising these models enables analysis of primary devices of sampling flowmeters.
- Five arts of mounting of point velocity sensor are presented and the formulae for distance of sensor from the pipe axis are given. This distance depends on Reynolds number and the kind of mathematical model taken for describing the velocity distribution.
- On the base of comparison of data introduced in table 1 and 2 it is visible, that critical position described in [1,2], for which the sensitivity factor is equal 1, is not suitable, because it changes with

Reynolds number changes. The critical position changes also with changes of velocity distribution shape (for example according to pipe roughness changes).

- The influence of roughness changes on the shape of velocity distribution (what takes place in many real situations - authors experiences in water supply systems confirmed it) can be so great, like the influence for Reynolds number changes in the ratio 1 : 1000 for hydraulically smooth pipe.
- The optimum position is the best because allows to achieve the smallest error of volume flow-rate measurement. If we have more information about the scope of measured value and about its value distribution it is possible to choose such position, which ensures the minimum error value according to the taken criterion.
- Rather simple centreline position causes small error connected with velocity sensor mounting (velocity distribution in the pipe centre is almost flat) but the value of sensitivity factor must be estimated for actual value of measuring volume flow-rate. The changes of sensitivity factor are rather great, for example for Reynolds number from $4x10^3$ to $17x10^6$ the sensitivity factor changes from 0,791 to 0,886.
- Critical position changes according to the changes of measuring value, and practically the nominal position must be chosen, and than the real sensitivity factor must be calculated with iterative method. For this calculation the Reynolds number must be calculated, it means the viscosity must be known (the fluid composition and temperature of the fluid).
- Introduced arts of placing of velocity sensor in one-point primary devices of sampling flowmeters and formulae for sensitivity factors allow to calculate the volume flow-rate and are the theoretical basis for further analysis of properties of primary devices of sampling flowmeters.

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