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HYDROSTATIC HEAD CORRECTION

W. Kolaczia
Section for Length, Area, Angle, Photometry, Static Volume
Bundesamt für Eich- und Vermessungswesen (BEV)
Arltgasse 35, A-1160 Wien, Austria

Abstract: The volume of a cylindrical storage tank with a vertical axis changes - besides of the influence of other quantities - with the hydrostatic pressure applied by the liquid stored in the tank. The formula of M. K. Sayadi [1] describes the influence of the liquid head on the expansion of storage tanks best having the handicap that it can be evaluated individually for each tank only by means of a numerical program because the explicit solution works only for a tank consisting of a single course. The author tried to meet two requirements: to give a formula for an explicit solution for tanks consisting of several courses and – to give a formula which leads to a result as accurate as that one obtained by numerical solution according to the formula of Sayadi. Both conditions are fulfilled by the formula derived.
Keywords: storage tanks, volume, measurement uncertainty

1 INTRODUCTION

The volume of storage tanks changes - besides of the influence of other quantities - with the hydrostatic pressure applied by the liquid stored in the tank. The resulting orders of magnitude are not to be neglected when assessing the uncertainty of the calibration measurement of the storage tank, in particular, if the calibration is carried out by the optical method in an empty tank. On one hand the paper presented shall help to minimize the component of the uncertainty of measurement caused by the expansion of the tank, on the other hand the calculation of the correction for expansion shall be carried out easily.

In analogy to Sayadi´s formula ([1], equation (25)) it can be seen how the author developed his formulae: the whole formula of Sayadi consists of 3 parts which can be interpreted as “BASIC MODEL”, “EXPANSION OF THE BASIC MODEL” (with regard to the vertical stress) and the “EXPANSION TO THE GENERAL SOLUTION” (with regard to the influence of the bottom of the storage tank). So the authors´ efforts will eventually lead to solutions valid for any given number of courses.

2 BASIC MODEL

The assumption of the basic model is that the liquid head leads to deformation of the shell in an horizontal layer. A mathematical apparatus shall be found describing the tangential stress of the shell in the case of a single course model and later with a model on the assumption of many courses.

2.1 Case of single course

Let us consider a tank in form of a circular cylinder with vertical axis and a radius R, liquid height L and plate thickness t. The relative change of the radius R with radial pressure P is given by equation

$$\frac{\Delta R}{R} = \frac{PR}{tE} \tag{1}$$

where E is Young´s modulus of elasticity. The radial pressure P in the height h, provided the tank is filled up to the height L, is a function of h. Therefore

$$P = (L-h)(r_L - r_C)g \tag{2}$$

where r_L is the density of the liquid stored in the tank, r_C is the density of the medium during the calibration procedure (air, water, etc.) und g is the acceleration due to gravity.

A tank in form of a circular cylinder with vertical axis deformed by the hydrostatic pressure can be considered as a rotational body and the change of the volume ΔV for the filling height L caused by the expansion of the tank can be calculated according to an equation, in which an expression for ΔR according to (1) is substituted

$$\Delta V = V - \pi \int_0^L \left(R - \frac{PR^2}{tE} \right)^2 dh \qquad (3)$$

Remove the expression R from within the brackets and put it as a constant R^2 in front of the integral. Then square the binominal function in the integral. If you neglect the quadratic terms of ΔR, the absolute value of the error resulting thereby is usually $\leq 5 \cdot 10^{-4}$ ΔV. As result of this and further more translations (like a substitution for P according to (2)) you will find at last

$$\Delta V \cong R^2 \pi \int_0^L \frac{2R(L-h)(\rho_L - \rho_C)g}{tE} dh \qquad (4)$$

Solving the integral you will get

$$\Delta V \cong \frac{R^3 \pi L^2 (\rho_L - \rho_C) g}{tE} \qquad (5)$$

For the relative volume of expansion ΔV/V applies

$$\frac{\Delta V}{V} \cong \frac{RL(\rho_L - \rho_C) g}{tE} \qquad (6)$$

2.2 Case of many courses

The assumption that a storage tank has the same radius and the same plate thickness for the whole height is not realistic; storage tanks in cylindrical shape with vertical axis are as a rule built up by several cylindrical sections with radius r_i, height of section Δh_i and plate thickness t_i. For the calculation of the correction of the expansion ΔV in equation (4) in the right-hand side the sum of the corrected volumina of the specific sections, that are relevant for the overall correction, is to be put in. Thus is

$$\Delta V \cong \frac{2\pi (\rho_L - \rho_C) g}{E} \sum_{i=1}^{n} \frac{r_i^3}{t_i} \int_{h_{i-1}}^{h_i} (h_n - h) dh \qquad (7)$$

where $h_o = 0$ and $h_n = L$. Solving the integrals leads to

$$\Delta V \cong \frac{2\pi (\rho_L - \rho_C) g}{E} \sum_{i=1}^{n} \frac{r_i^3}{t_i} \left[h_n (h_i - h_{i-1}) - \left(\frac{h_i^2}{2} - \frac{h_{i-1}^2}{2} \right) \right] \qquad (8)$$

Because of $\frac{h_i^2}{2} - \frac{h_{i-1}^2}{2} = (h_i - h_{i-1}) \frac{h_i + h_{i-1}}{2}$ and $\Delta h_i = h_i - h_{i-1}$ we get

$$\Delta V \cong \frac{2\pi (\rho_L - \rho_C) g}{E} \sum_{i=1}^{n} \frac{r_i^3}{t_i} \Delta h_i \left(h_n - \frac{h_i + h_{i-1}}{2} \right) \qquad (9)$$

Substituting $r_i = R$ and $t_i = t$ for all i and $h_n = L$ in equation (9) leads to equation (5).

3 EXPANDED MODEL

The formulae for the calculation of the correction for the expansion mentioned above are based on a physical-mathematical model that considers the tangential stress of the shell by the radial pressure P in a horizontal plane due to the hydrostatic pressure, caused by the liquid head but not the stress in the vertical direction resulting from it. The corrections of the expansion calculated according to these formulae are subject to an error which contributes to the measurement uncertainty while calibrating the storage tank. If this measurement uncertainty shall be minimized, the model for the calculation of the corrections for the expansion has to be modified in an appropriate way. An approach is the model used by Sayadi in his paper which, on the base of Hooke´s law, considers the stress of the shell in all three spatial directions due to hydrostatic pressure.

3.1 Case of a single course

For the simple case of a storage tank with height H, radius R and plate thickness t an explicit formula for the calculation of the relative volume of expansion $\Delta V/V$ is given in [1] by equation (25), a simplification of this formula is given by equation (27). For the calculation of the volume of expansion of a storage tank assembled by several cylindrical sections with radius r_i, height of the section Δh_i and plate thickness t_i an implicit formula is given with equation (32), the solution of which only could be done by a program for the numerical computation of the integral in (32) separate for each tank.

Studying [1] shows that the complexity of the mathematical model grows by each improvement of the physical model very quickly and complicates the understanding and the applicability. On the other hand a simplification with a sufficiently good approximation is at hand as equation (27) shows. In the following we shall try to connect the results of [1] with those of the paper presented in a way so that a sufficiently good approximation is guaranteed despite the simplicity of the mathematical model.

The comparison of equation (27) in [1] and equation (6) of the paper presented shows that the influence of the vertical component of the stress of the shell due to hydrostatic pressure can be described by the expression -μ/2, where μ is the Poisson ratio (for steel ≈ 0,30). Taking into account the vertical component in (5) you get

$$\Delta V \cong \frac{R^3 p\, L^2 (r_L - r_C)\, g}{tE}\left(1 - \frac{m}{2}\right) \tag{10}$$

being the equation for the calculation of the volume of expansion. In order to derive equation (10) in the same way as it was done with equation (5), equation (1) has to be modified as follows:

$$\frac{\Delta R}{R} = \frac{PR}{tE}\left(1 - \frac{m}{2}\right) \tag{11}$$

If you substitute in (3) for ΔR according to (11), you get

$$\Delta V = V - p \int_0^L \left[R - \frac{PR^2}{tE}\left(1 - \frac{m}{2}\right)\right]^2 dh \tag{12}$$

You can remove R from the term within the brackets and put it as a constant in R^2 front of the integral. By squaring the binominal expression in the integral and by eliminating the quadratic term and substituting according to (2) you get after reducing and transforming correspondingly

$$\Delta V \cong (2 - m)\, R^3 p \int_0^L \frac{(L-h)(r_L - r_C)\, g}{tE}\, dh \tag{13}$$

Solving the integral leads to equation (10).

3.2 Case of many courses

For a storage tank assembled by n cylindrical sections with radius r_i, height of the section Δh_i and plate thickness t_i the correction for the volume of expansion ΔV can be calculated consequently according to the following equation:

$$\Delta V \cong \frac{(2 - m)p\ (r_L - r_C)\ g}{E} \sum_{i=1}^{n} \frac{r_i^3}{t_i} \int_{h_{i-1}}^{h_i} (h_n - h)\, dh \tag{14}$$

By solving the integral you get after reducing and transforming correspondingly

$$\Delta V \cong \left(1 - \frac{m}{2}\right) \frac{2p\ (r_L - r_C)\ g}{E} \sum_{i=1}^{n} \frac{r_i^3}{t_i} \Delta h_i \left(h_n - \frac{h_i + h_{i-1}}{2}\right) \tag{15}$$

4 GENERAL SOLUTION

Both models introduced give a maximum ΔR for h = 0; because of the stable bottom of the storage tank which prevents any expansion in the vicinity of the bottom of the storage tank ΔR = 0 applies for h = 0. Therefore the expansion of the storage tank calculated according to equation (15) has to be corrected. As you can see from the paper of Sayadi the influence of the bottom of the storage tank on the expansion volume of the first course can be described by a factor k:

$$k = \left[\left(1 + \frac{1}{H}\right)^2 - \frac{H\left(e^{2H} + 1 + \sin 2H\right) + 2\,(H+1)\cos^2 H}{H^2\left(\cos^2 H + \cosh^2 H\right)} \right] \tag{16}$$

whereas H is defined by the equation

$$H = h\left[\frac{3\left(1 - m^2\right)}{t^2 r^2}\right]^{\frac{1}{4}} \tag{17}$$

with the tank filling height h, inner radius r and plate thickness t.

The correction factor k is dimensionless and determined completely by the parameter H and therefore applicable for all storage tanks with one course. Of course, if we are in the position to modify equation (17) in such a way that H is defined for the case of a storage tank consisting of several courses we have found a general solution. For an easy understanding imagine several courses not piled up but put side by side. For each of these courses equation (17) must apply; Δh_1 being the height of the first course, Δh_2 the height of the second course, Δh_i the height of the i^{th} course:

$$H_i = \Delta h_i\left[\frac{3\left(1 - m^2\right)}{t_i^2 r_i^2}\right]^{\frac{1}{4}} \tag{18}$$

Piling up these courses, $H = \sum_{i=1}^{n} H_i$, applies, and by (18) we will find the equation

$$H = \sum_{i=1}^{n} \Delta h_i\left[\frac{3\left(1 - m^2\right)}{t_i^2 r_i^2}\right]^{\frac{1}{4}} \tag{19}$$

The complete equation which takes the influence of the bottom of the storage tank also into account is

$$\Delta V \cong \left[\left(1+\frac{1}{H}\right)^2 - \frac{H\left(e^{2H}+1+\sin 2H\right)+2\,(H+1)\cos^2 H}{H^2\left(\cos^2 H+\cosh^2 H\right)}\right]\left(1-\frac{m}{2}\right)\frac{2p(r_L - r_C)g}{E}\sum_{i=1}^{n}\frac{r_i^3}{t_i}\Delta h_i\left(h_n - \frac{h_i + h_{i-1}}{2}\right) \tag{20}$$

whereas H fits equation (19). Substituting r_i = R and t_i = t for all i and h_n = h in equation (20) leads to an equation which is equivalent to the equation (25) in [1].

5 CONCLUSION

The formula derived is not more complicated than that one given by Sayadi in [1] but it is exact enough taking into account the vertical component of the stress of the shell caused by radial pressure and the influence of the bottom of the storage tank consisting of several courses. By the formula derived it is possible to calculate on the one hand the correction for the expansion of the storage tank and on the other hand to minimize the component of the measurement uncertainty caused by the expansion of the storage tank while calibrating cylindrical storage tanks with vertical axis in accordance with [2], [3] and [4].

Figure 1 shows an example of how big the deviations of the expansion volumina can be related to the respective volumina of the storage tanks calculated according to formulae (9) and (15):

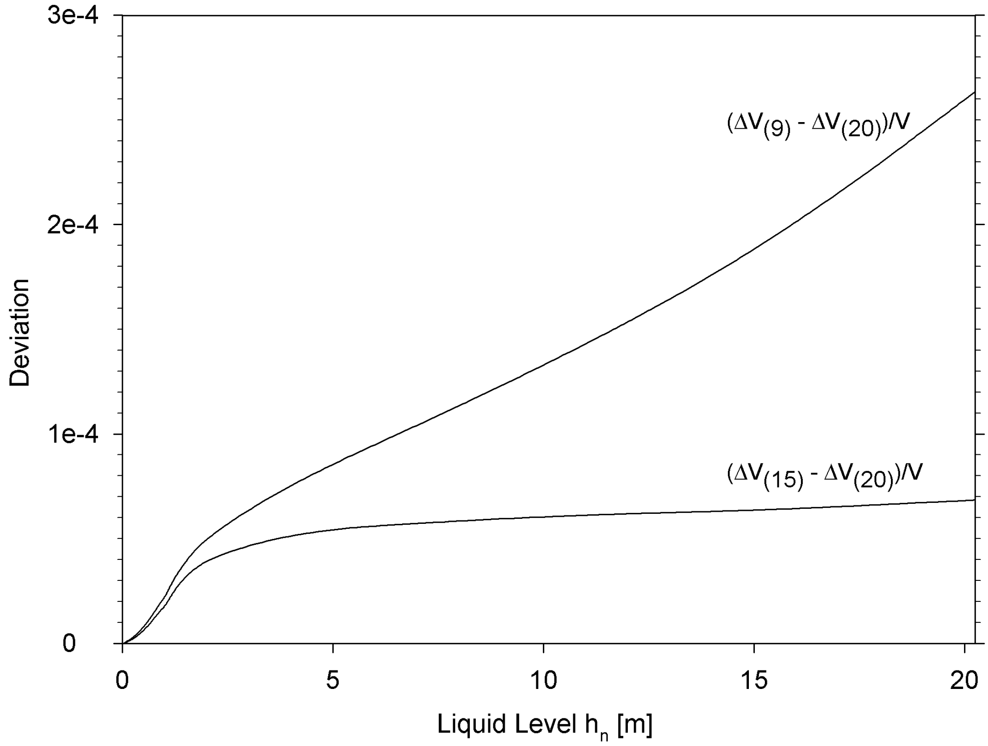

Figure 1. Example calculation (see text for details)

This example is based on a storage tank with radius r = 35,5 m, consisting of 9 courses with the plate heights Δh_i = 2,3 m whereas the plate thickness t_i varies from 35,8 mm (t_1) to 12,4 mm (t_9); the maximum filling height h_n is 20,5 m. This example is a quite extreme one (capacity of the storage tank more than 80 000 m^3) but it will be helpful to estimate the maximum relative deviation in the upper range since smaller storage tanks show only smaller relative deviations, of course. In figure 1 the relative deviation for any filling height can be seen: the maximum relative deviation of the expansion volumina calculated by equation (9) and (20), respectively, is $26{,}3 \cdot 10^{-5}$ and that one calculated by equation (15) and (20), respectively, is $6{,}9 \cdot 10^{-5}$. Therefore, in case the highest possible accuracy is stipulated, it is very advisable to apply equation (20) for calculating the expansion of a cylindrical storage tank with a vertical axis due to liquid head.

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AUTHOR: Dr. Wilhelm KOLACZIA, Section E23: Length, Area, Angle, Photometry, Static Volume, Bundesamt für Eich- und Vermessungswesen (BEV), Arltgasse 35, A-1160 Vienna, Austria
Phone Int. +43 1 49 110 509, Fax Int. +43 1 49 20 875, E-mail: w.kolaczia@bev-eich.gv.at