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IMPROVED RECURSIVE ALGORITHM FOR PERIODIC SIGNALS

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Abstract: An established robust recursive algorithm for the identification of periodic signals is further developed. Contemporary algorithms for recursive harmonic estimates are restricted in that, once a harmonic value is estimated after one cycle, no further improvement is attained. It is demonstrated that higher order algorithms offer data for the assiduous improvement to the calculated harmonic evaluation. Second and third order improvements to the algorithm are presented. Data smoothing of the recursive frequency analysis estimates allows marked convergence to the authentic value. The enhanced procedure illustrates its capacity to resolve the parameters of an unknown signal. The paper demonstrates that the algorithm is robust in that better convergence is still achieved in the presence of a noise-contaminated signal.

Keywords: Identification, Recursive Algorithm

1 INTRODUCTION

Considering the character of response to systems under interrogation, it is necessary to analyse a system's frequency response using signal-processing procedures. This frequency analysis technique, which is normally an off line identification approach, is based upon the methods of discrete Fourier analysis. A practical recursive algorithm based on periodic multi-frequency signals has been developed for recursive frequency measurement [1]. This algorithm is elementary, expeditious and simple to implement for on line, in situ measurement. It has been successfully applied to measure the spectrum of the ternary multi-frequency sequence (MTS) and to test an electrical resistance furnace using a binary multifrequency sequence (MBS) excitation signal [2]. Various cascaded digital filters have improved the smoothness and the speed of convergence of the algorithm. The quality of the algorithm, which has been verified by statistical theory, has been investigated by simulation when applied with various types of filter [3].

The higher order techniques, Trapezium and Simpson's methods, enjoy the characteristics of the fundamental (Euler's) method and permit the evaluation of the parameters for an unknown signal. Further, the higher order renditions advance the opportunity to decrease the error in the estimates by 25%. Mathematical proof, echoed by simulation, shows that parameter estimates by the algorithm are unbiased and consistent. The robustness of the recursive DFT has also been studied. The paper presents a summary of these developments in recursive frequency analysis.

2 EULER'S METHOD

For a sampled signal, an interval data record in noise, x_m, is collated. The discretised period of the fundamental harmonic extends over N data points. The inter-sample time is denoted by T_s. The complex coefficient of the k^{th} harmonic of the Fourier series of x_m, which has been evaluated between the r^{th} to the $(r + N + 1)^{th}$ data points is given the notation $X_{m,r}(\omega_k)$. This coefficient is usually calculated using the customary DFT. In order to overcome any phase inconsistencies, the samples are actually taken at times $x_{m,r}$ (n + 0.5). This leads to the difference between the coefficients for $X_{m,r}(\omega_k)$, and $X_{m,r+1}(\omega_k)$. That is, the coefficient for the k^{th} harmonic is updated with each sample. These samples are then used to supply the core information necessary for the algorithm. The authors have demonstrated that after some rearrangement the difference between these two approximates, generate the recursive algorithm given by equation (1).

$$X_{m,r+1}(w_k) \quad = \quad X_{m,r}(w_k) - \frac{1}{N}\left[x_m(r+0.5) - x_m(r+0.5+N)\right]e^{-j\frac{2p\,k}{N}(r+0.5)} \qquad (1)$$

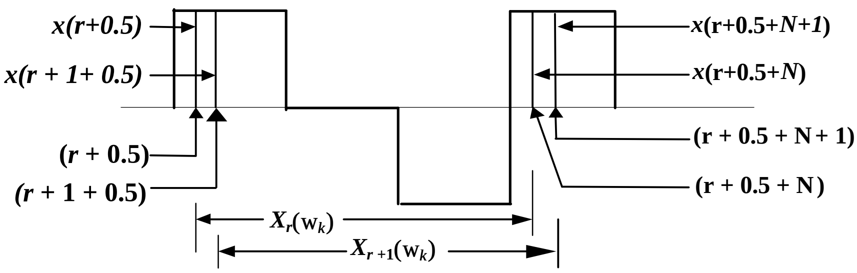

Figure 1. The discrete interval data record for the S3T1 Multifrequency Ternary Signal (MTS).

The fundamental kernel of the DFT calculation is known as Euler's method. Figure 1. Illustrates the method as used for the S3T1 Multifrequency Ternary Signal (MTS). This signal, as others in the multifrequency family, are particularly favoured for their qualities of providing detailed specific information for system interrogation, and their accommodation to clear analysis [2].

Before any improvement to this algorithm can be elucidated, its origins must be delineated to view if the method can be improved upon. Note that from figure 1, that the current value of the calculated harmonic is updated in real time as each sampled data point is recorded. As the algorithm complexity is increased then the inter-sample time T_s must be considered with respect to microprocessor speed. Because of the modest size of the algorithm, even in the case of the highest order renditions, the limiting constraints on the method are usually the transducers and recording instrumentation.

2.1 Euler's technique

The simplest form of integral approximation is generated by the use of Riemann summation, figure 2. This subdivides the plot into vertical strips, each of length T_s. The value of the sampled function at t = nT_s is recorded as $x(nT_s)$ and is multiplied by T_s to form the element of the integral due to this point, figure 2. Therefore the integral (I), over the whole period is given by

$$I = \int_0^T f(t)dt \approx \sum_0^{N-1} x(nT_s)T_s \qquad (2)$$

Where $x(nT_s)$ is the value of the sampled waveform at the discrete time $t = nT_s$. Therefore, applying the DFT to the sampled data gives the complex amplitude for the k^{th} harmonic (3), where $X(\omega_k)$ is the numerical approximation to the true value C_k.

$$C_k = \frac{1}{T}\int_0^T f(t)e^{-jkw_0 t}dt \approx X(w_k) = \frac{1}{N}\sum_0^{N-1} x(nT_s)e^{-j\frac{2p\,k}{N}n} \qquad (3)$$

Note from figure 2, that because it is a discrete method, there is the potential to either underestimate

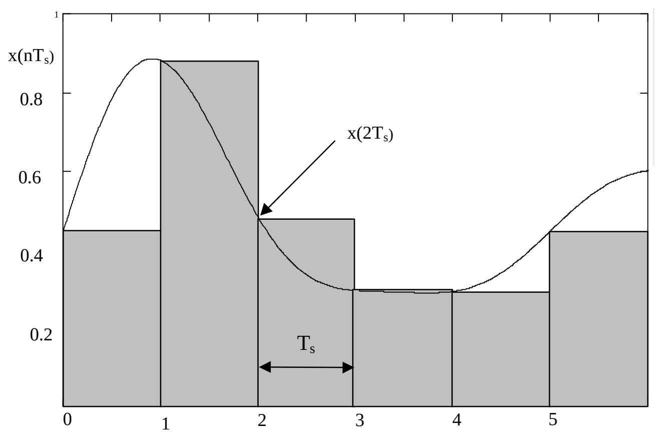

Figure 2. Euler's Method, Integral is approximated by the sum the rectangular areas.

or overestimate the value of the harmonic. However, largely by the use of MBS and MTS signals this propensity for error is severely weakened. A more severe test of the vigour of the algorithm is to engage a more generic, undulating, form of periodic signal. This test has also been applied to the algorithms.

When a system is sampled, an array, or data record of the sample values is built up. If the n[th] sample is labelled x_n, then the complex harmonic amplitude can be rewritten as $X(w_k) \;=\; \frac{1}{N}\sum_{0}^{N-1} x_n\, e^{-j\frac{2p\,k}{N}n}$. Now for a general point r in the sampling process the complex harmonic has the generic description $X_r(w_k) \;=\; \frac{1}{N}\sum_{n=r}^{N-1+r} x_n\, e^{-j\frac{2p\,k}{N}n}$, and for the next point in the sequence, r + 1, the complex harmonic has the generic description

$$X_{r+1}(w_k) \;=\; \frac{1}{N}\sum_{n=r+1}^{N+r} x_n\, e^{-j\frac{2p\,k}{N}n} \;=\; \frac{1}{N}\sum_{n=1}^{N} x_{n+r}\, e^{-j\frac{2p\,k}{N}(n+r)} \tag{4}$$

The difference between the two sequentially calculated values is given by equation (5).

$$X_{r+1}(w_k) \;-\; X_r(w_k) \;=\; \frac{1}{N}\left\{ x_{n+r}e^{-j\frac{2p\,k}{N}(N+r)} - x_r e^{-j\frac{2p\,k}{N}r} \right\} \tag{5}$$

but $e^{-j2pk} \;=\; 1$ whenever $k \in$ **Z**, this permits equation (5) to be simplified and rearranged to give

$$X_{r+1}(w_k) \;=\; X_r(w_k) \;+\; \frac{1}{N}\left\{ x_{n+r} - x_r \right\} e^{-j\frac{2p\,k}{N}r} \tag{6}$$

For the case of a real sampled signal, contaminated by noise, an interval data record in noise, x_m, is collated, where x_m is the sum of the original signal x_n and a tainted sequence x_{noise}. The complex coefficient of the k[th] harmonic of the Fourier series of x_m, which has been evaluated between the r[th] to the (r + N + 1)[th] data points is now given the notation $X_{m,r}(\omega_k)$.

$$X_{m,r}(w_k) \;=\; \frac{1}{N}\sum_{n=r}^{N-1+r} x_m(n)\, e^{-j\frac{2p\,k}{N}n} \;=\; \frac{1}{N}\sum_{n=0}^{N-1} x_m(n+r)\, e^{-j\frac{2p\,k}{N}(n+r)} \tag{7}$$

In order to overcome any phase inconsistencies, the samples are taken at times x_m,(r+n+0.5). This leads to the coefficients $X_{m,r}(\omega_k)$ being evaluated as,

$$X_{m,r}(w_k) \;=\; \frac{1}{N}\sum_{n=0}^{N-1} x_m(n+r+0.5)\, e^{-j\frac{2p\,k}{N}(n+r+0.5)} \tag{8}$$

and for the next in line calculation, the (r+1)[th], is given by

$$X_{m,r+1}(w_k) \;=\; \frac{1}{N}\sum_{n=1}^{N} x_m(n+r+0.5)\, e^{-j\frac{2p\,k}{N}(n+r+0.5)} \tag{9}$$

The difference between equations (8) and (9) generates the recursive algorithm.

$$X_{m,r+1}(w_k) \;=\; X_{m,r}(w_k) - \frac{1}{N}\left[x_m(r+0.5) - x_m(r+0.5+N) \right] e^{-j\frac{2p\,k}{N}(r+0.5)} \tag{1 bis}$$

3 TRAPEZIUM METHOD

The next improvement in the armoury of numerical integration methods is called the Trapezium method of integration, figure 3. It is immediately apparent from figure 3 that the potential level of error in the estimation is decreased by the use of this technique. This method subdivides the plot of the sampled data points into trapezoidal strips, each of length T_s equal to the inter-sample period. At a given point r the height of the strip is approximated by [x(r) + x(r+1)]/2. The area of the strip at r is therewith approximated by T_s[x(r) + x(r+1)]/2. Now applying the trapezium method to the sampled sequence for the DFT gives.

$$X_{m,r}(w_k) = \frac{1}{2N}\left[x(0)e^{-j\frac{2p\,k}{N}(0.5)} + 2\sum_{n=1}^{N} x(nT_s)e^{-j\frac{2p\,k}{N}(n+0.5)} + x([N-1]T_s)e^{-j\frac{2p\,k}{N}(N-1+0.5)}\right] \quad (10)$$

Following an analysis similar to that employed in Euler's method; two adjacent sample points in a contaminated signal are considered. The generated recursion is given by equation (11). The quality of the second order version of the algorithm is reported in section 5.

$$X_{m,r+1}(w_k) = X_{m,r}(w_k) - \frac{1}{2N}\left[\begin{array}{l}\{x_m(N+r-0.5) - x_m(N+r+0.5)\}e^{-j\frac{2p\,k}{N}(r-1)} + \\ + x_m(N+r-1)e^{-j\frac{2p\,k}{N}(r-1)} - x_m(r+1)e^{-j\frac{2p\,k}{N}(r+1)}\end{array}\right] \quad (11)$$

4 SIMPSON'S METHOD

This time a quadratic approximation is used to interpolate between three adjacent sample points. The generalised DFT for the k^{th} harmonic is given by equation (12). Following the same type of procedure as before, this leads to a recursion relation similar in form to Euler's and the Trapezium equation (13). Simpson's method is derived by interpolating equally spaced data points, (r-1, x_{r-1}), (r, x_r), and (r +1, x_{r+1}), deriving a quadratic polynomial from these and then integrating this quadratic. Since it can allow for the curvature of the graph of the sampled function, Simpson's method gives much more accurate results than Euler's method, or the trapezoidal method, figure 4. Improvement is noticeable particularly in noisy signals. Results of all methods are compared in section 5.

$$X_{m,r}(w_k) = \frac{1}{3N}\left[\begin{array}{l} x(0)e^{-j\frac{2p\,k}{N}(0.5)} + 4\sum_{n=1}^{\frac{N-1}{2}} x(2n-1+0.5)e^{-j\frac{2p\,k}{N}(2n-1+0.5)} + \\ + 2\sum_{n=1}^{\frac{N-3}{2}} x(2n+0.5)e^{-j\frac{2p\,k}{N}(2n+0.5)} + x(N-1)e^{-j\frac{2p\,k}{N}(N-1+0.5)}\end{array}\right] \quad (12)$$

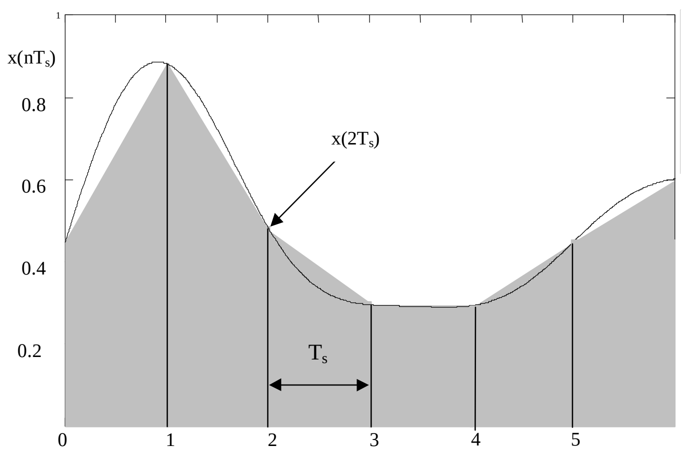

Figure 3. Trapezium Method, Integral is approximated by the sum of trapezoidal areas.

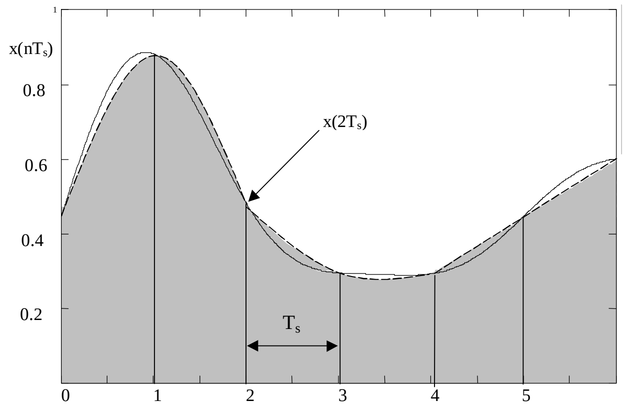

Figure 4. Simpson's Method, Integral is approximated by the sum of quadratic areas.

$$X_{m,r+1}(w_k) = X_{m,r}(w_k) - \frac{1}{3N}\begin{bmatrix} \{5x_m(N+r+0.5) - x_m(r+0.5)\}e^{-j\frac{2p\,k}{N}(r+0.5)} + \\ + x_m(N+r-0.5)e^{-j\frac{2p\,k}{N}(r-0.5)} + \\ -3x_m(r+1+0.5)e^{-j\frac{2p\,k}{N}(r+1+0.5)} + \\ -2x_m(r+2+0.5)e^{-j\frac{2p\,k}{N}(r+2+0.5)} \end{bmatrix} \tag{13}$$

5 RESULTS

Experimental results for the algorithms are displayed in figures 5 and 6. Errors in the calculates are tabulated for comparison in table 1. In figure 5(a), the convergence of Euler's method is shown. Figure 5(b) contrasts Euler to the Trapezium. Figure 5(c) contrast all three. Figure 5(d) contrasts all three in detail. These are repeated respectively for the noise-contaminated case in figure 6.

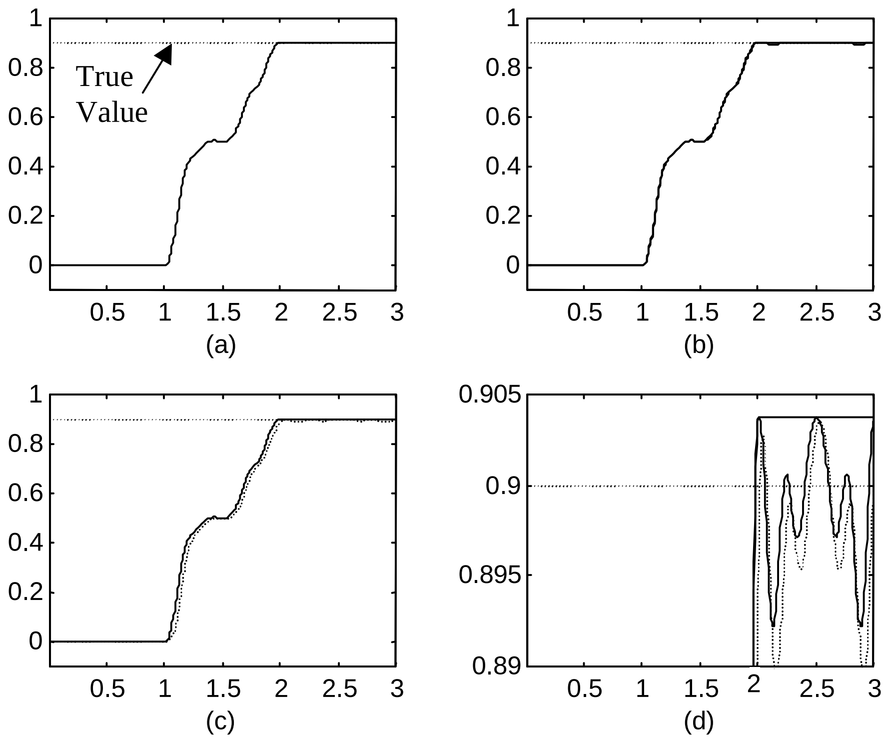

Figure 5. The calculated harmonics compared. Recursion applied to noisefree signal. Abscissa, time in seconds. Ordinate, amplitude of harmonic. Seperation clear only in detail, in subfigure (d).

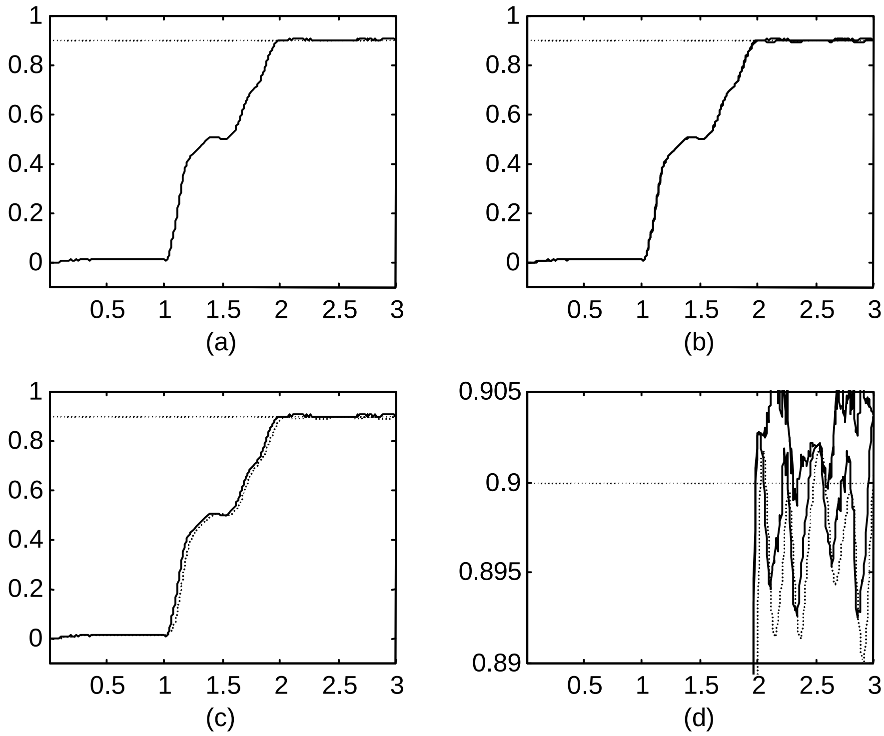

Figure 6. The calculated harmonics compared. Recursion applied to a noisy signal with variance of 0.2. Subfigure (d), all three methods compared in detail. Under noisy conditions higher order methods are more stable.

Table 1. Percentage errors compared. Second and third order methods show valuable improvement. When a contaminated signal, carrying noise with variance of 0.2 is monitored, the strengths of higher order methods are clearly evident.

Signal condition	Euler / % error	Trapezium / % error	Simpsons / % error
ST31 Pure signal	0.3746	0.3541	0.0146
ST31 Variance = 0.2	0.3786	0.3648	0.0467
Generic periodic Pure signal	5.4461	5.4122	4.5846
Generic periodic Variance = 0.2	4.2276	4.1964	3.3976

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