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A DIDACTICAL DEVICE FOR INVESTIGATIONS ON A QDW SYSTEM

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Abstract: Very hard studies are often accomplished in order to deeply understand the features of complex system. Mathematical models represent a useful approach for developing theoretical considerations. Anyway experimental set up, allowing heuristic confirmation of the results obtained via the theoretical approach must be considered a fundamental step toward a full understanding of the considered system. In this work a quite innovative environment for experimental investigation on the Brownian Motion system is proposed. A virtual instrument allowing easy of use and flexibility to the collection of both results and information on the system state, has been implemented.
Keywords: Virtual instrument, Brownian Motion, analog device

1 INTRODUCTION

The Brownian Motion model is used in many physical branches as a prototype for the comprehension of several natural phenomena [1-5]. Several studies have emphasised the features of this system and its complexity due to the deep sensitivity to the structural parameters [3]. The term Brownian Motion is used to describe the movement of a particle in a liquid, subjected to collisions and other forces. Macroscopically, the position $x(t)$ of the particle can be modelled as a stochastic process satisfying a second-order differential equation:

$$m\,\ddot{x}(t) + g\,\dot{x}(t) + \dot{V}(x) = e(t) + f(t) \quad (1)$$

where $e(t)$ is the collision force, m is the mass of the particle, g is the coefficient of friction, $f(t)$ is a generic forcing term and $V(x)$ is a general potential function. On a macroscopic scale, the process $e(t)$ can be viewed as normal white noise with zero mean and power spectrum $S(w)=2kTf$, where T is the absolute temperature of the medium and k is the Boltzman constant [6].

Very interesting behaviour of the system (1) has been observed when $V(x)$ is a Quartic Double Well potential (QDW) [4, 5]:

$$V(x) = -a\frac{x^2}{2} + b\frac{x^4}{2} \qquad \text{with } a>0 \quad (2)$$

The minima of V(x) are located in

$$x_+=(a/b)^{1/2} \text{ and } x_-=-(a/b)^{1/2} \quad (3)$$

and are separated by a potential barrier with amplitude

$$DV=a^2/(4b). \quad (4)$$

The characteristic frequencies of the system are

$$\omega_0=(2a)^{1/2} \text{ and } \omega_b=(a)^{1/2} \quad (5)$$

The system (1), with the potential $V(x)$ defined in (2), has been studied for many years and still represents a topic of considerable scientific interest. In fact, the model has been used as a mathematical model to represent a number of natural phenomena. For instance, in the classical sense, it represents the movement of a steel beam subjected to electromagnetic stress held pinned to fixed supports. The fixed supports provide the system with non-linear stiffness that can be expressed by the term bx^3. In this case, the physical interpretation of the sign of the parameter is as follows: if *a>0*, the term in *x* represents a resistive elastic torque, which is the case of *QDW*, while if *a<0* it represents an active elastic torque.

For this and any other system of the same or greater complexity, it can be impossible to find out an analytical solution; however, by using both numerical simulation and suitable analog device its behaviour can be observed.

In section four a virtual instrument developing a user interface for observations on a QDW system is proposed.

2 SOME NOTES ON THE QDW SYSTEM

In the presence of a strong periodic forcing signal *f(t)* (in case of *e(t)*=0), the potential *V(x)* is tilted back and forth raising and lowering successively the potential barrier of the right and the left well. As a consequence the particle rolls periodically from one potential well into the other one with the same frequency of the forcing signal. In presence of a weak signal the particle lays in a narrow range (attractor) of the state which is selected by the initial conditions. In Figure 1 the shape of *V(x)* and the state diagrams in case of weak and strong forcing signal are reported.

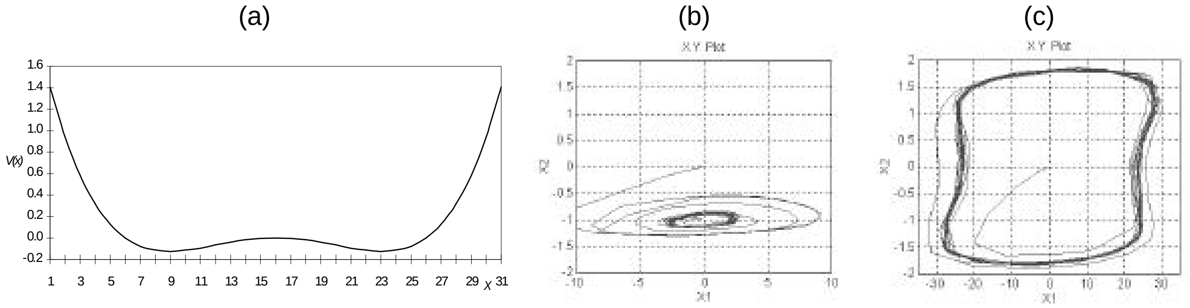

Figure 1. The shape of *V(x)* and the state diagrams when a weak or a strong forcing signal is applied.

The system being investigated strongly depends on parametric variations. As the parameters vary, the system modifies its behaviour, passing from periodic states, characterised by limit cycles, to chaotic behaviour [3].

In the considered case, the parameters will be chosen in such a way that the system presents a variety of non-linear dynamics and, according to the value of the forcing signal, its state remains confined in the proximity of a minimum or oscillates between the two minima.

For the sake of convenience the system can be rewritten as follows

$$\ddot{x} = -g\dot{x} + ax - bx^3 + f \tag{6}$$

By simple calculation a system of two first-order equations can be obtained

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -gx_2 + ax_1 - bx_1{}^3 + f \end{cases} \tag{7}$$

The system typically has dynamics at a very low frequency [3-5]. As one of our aims is to develop an analog simulation environment, we chose temporal scaling of the linear part of the system to vary the system bandwidth. By adopting this procedure the following system can be obtained

$$\ddot{x} / k^2 = \frac{-g}{k}\dot{x} + ax + f \tag{8}$$

Hence

$$\begin{cases} \dot{x}_1 / k^2 = x_2 \\ \dot{x}_2 = -gkx_2 + ax_1 + f \end{cases} \tag{9}$$

Figure 2 shows the behaviour of the system when subjected to forcing signals with different amplitudes. With amplitudes below the system threshold the system makes no state transition and the state x_1 remains confined around x_-. This is evident from the trend of the state variable x_1 and the state diagrams in Figure 2a. The reason why the system state falls into this attractor is to be sought in the value of the initial conditions of the system. For these simulations the initial conditions were set in the attraction domain of x_-.

As the amplitude of the forcing signal grows up, a gradual passage from a strange to a periodic regime is observed, the latter having a frequency equal to that of the forcing signal, as illustrated in Figures 2b and 2c.

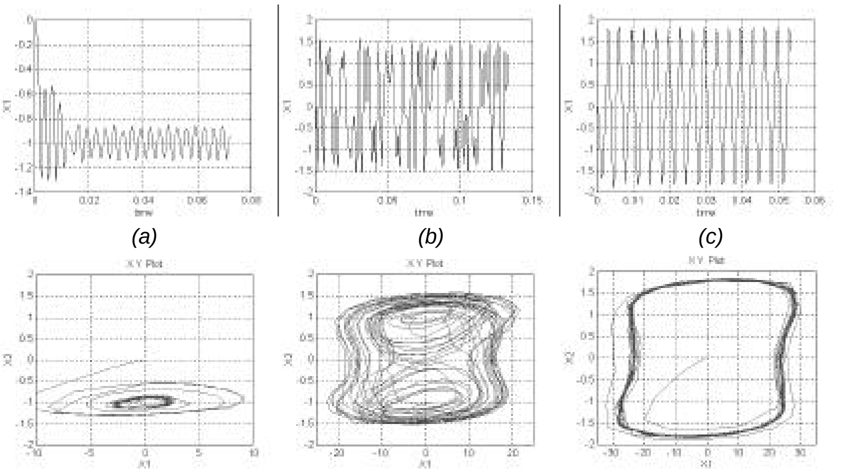

Figure 2. Temporal trend and system state plot of x_1, with a forcing signal amplitude of 0.1 V, 0.3 V, 0.5 V.

Besides observing the behaviour of the system with varying forcing signal amplitudes, it is interesting to investigate the system's response when the parameters *a* and *b* are varied. Simulations were therefore run varying the parameters *A*, *n*, *a* and *b*. The intervals used in each simulation were as follows

A=[0.1, 0.5] V step *0.05* V
n=[0.4, 2] rad/s step *0.4* rad/s
a=[0.1, 6] step *0.6*
b=[0.1, 6] *step 0.6.*

In the first set of simulations *A, n* and *a* were varied and *b* was chosen in such a way that *DV* remained constant and equal to 0.1. By varing the parameter *a* the modification of the inter-well distance is then obtained.

The aim of these simulations was to see how it was possible to reach an optimal switching condition (emphasising the action of the forcing signal) and how this is linked to the choice of an optimal value for the inter-well distance, once *A* and *n* (and therefore the forcing signal) are fixed.

As an example, Figure 3 shows the *QDW* output signal spectra for two pairs of values (*A, n*) with varying values for the parameter *a*. As can be seen, the family of spectra presents a peak corresponding to the frequency of the periodic forcing signal.

In order to identify the value of the parameter *a* with which the optimal switching condition occurs, being fixed *A* and *n*, a selection algorithm was used, based on the following criteria:

1) with a certain value of the parameter *a*, if the system output switches between the two states x_- and x_+, the *a* value is given an index *q=1*; otherwise *q=0*;

2) if the output signal spectrum has a maximum corresponding to the frequency of the forcing signal and if *q=1* that value of *a* becomes a candidate and an index *n* associated with each pair (*A, n*) is increased by one unit. The value of the index *n* associated with each pair (*A, n*) indicates the number of *a* values that make the system capable of switching between one state and the other one at the frequency of the forcing signal;

3) for each couple (*A, n*), the index $J=M_2/(M_1+M_2+M_3)$ is calculated for the values of *a* being candidate, where M_2 represents the maximum value of the spectrum at the frequency of the forcing signal, and M_1 and M_3 are the upper value bounds closest to it. The index *J* therefore weights the amplitude of the maximum point of the spectrum with reference to the other peaks, thus showing the system's capacity to outline the presence of the periodic input component;

4) for each couple (*A, n*), the *a* value at which the index *J* is maximum is calculated and taken as the optimal parameter for that couple.

The optimal value of the parameter *a* for each amplitude *A* and pulse *n* are given in the map in Figure 3c. An interesting property is observed: as the frequency increases, when the threshold is kept constant, the optimal value for *a* increases. It can be demonstrated that this is correlated with a reduction of the distance between the two wells at increasing frequency values. For the threshold to remain constant and equal to *K*, in fact, the parameter *b* has to take the value $b=a^2/(4K)$; the distance between the two wells therefore becomes $d=2(a/b)^{1/2}=4(K/a)^{1/2}$, so as *a* increases the distance *d* decreases.

It is also observed that when the frequency is kept constant, as the forcing signal amplitude *A* increases the optimal value for *a* decreases. It can be demonstrated that this corresponds to an increase of the distance between the two wells when the amplitude of the periodic forcing signal increases. Similar consideration can be developed for increasing value of the potential barrier *DV* controlled by changing the parameters *a* and *b* in a way to keep constant the distance between the wells.

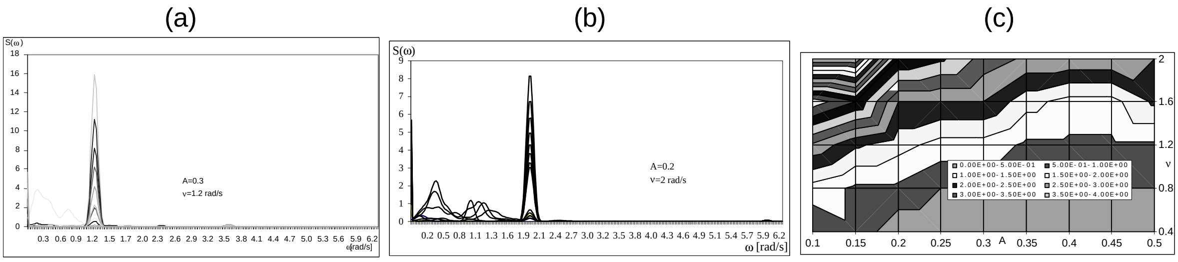

Figure 3. Output signal spectra, for different values of the parameter *a*. The curves show a peak correspondent to the forcing signal frequency. Map of the optimal values of *a.*

3 ANALOG IMPLEMENTATION OF QDW AND THE MONITORING SYSTEM

The analog implementation of the system is based on the circuit scheme shown in Figure 4. From the equations at the circuit nodes it is:

$$(u-Vc_1)/R_1 = C_1\dot{V}c_1 + Vc_1/R_3 + (Vc_1+Vc_2)/R_2 \qquad (10)$$

for the first node, and

$$Vc_1/R_3 = -C_2\dot{V}c_2$$

for the second node, where: $u = f + Vc_2^3$

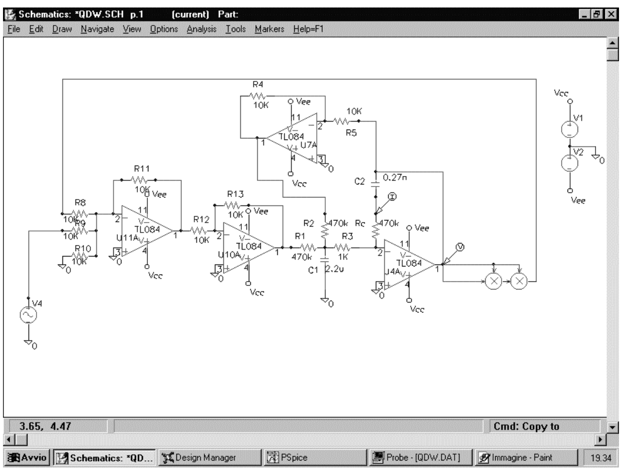

Figure 4. Circuit scheme for the analog implementation of the QDW system.

Rewriting these equations it is:

$$\dot{V}c_1 = -Vc_1(1/R_1C_1 + 1/R_2C_1 + 1/R_3C_1) - Vc_2 / R_2C_1 + u / R_1C_1$$
$$R_3C_2\dot{V}c_2 = -Vc_1 \qquad (11)$$

Comparing the system (11) with the system (9) the following correspondence between the circuit parameters and the system parameters

$$k^2 = 1/R_3C_2\,;\ gk = 1/R_1C_1 + 1/R_2C_1 + 1/R_3C_1$$
$$a = 1/R_2C_1\,;\ b = 1/R_1C_1 \qquad (12)$$

For a set of values commonly assigned in the literature to the system parameters

$g = 0.236; a = b = 0.967$, it is

$R_1 = R_2 = 470k\Omega$; $R_3 = 1k\Omega$; $C_1 = 2.2mF; C_2 = 270\,pF$; $(Rc = 470k\Omega)$

4 A VIRTUAL INSTRUMENT FOR INVESTIGATION ON THE QDW SYSTEM

To assess the effects of parametric variations on the behaviour of the system, it is useful to adopt a strategy that will allow us to intervene interactively on the parameters of interest and observe the system's response. The parameters taken into consideration are those modulating the shape of the potential *V(x)* and the product *gK* of the friction coefficient and the scaling factor.

To vary these parameters three potentiometers can be used which, acting on the values of the resistance R_1, R_2 and R_3, allow the shape of the potential to be modelled, i.e. the distance between the two wells and the height of the barrier. From Equations (12) it can be seen that the parameter *a* depends on R_2, parameter *b* on R_1 and K on R_3. Finally, it can be hypothesised (with suitable simplifications) that the parameter *gK* depends exclusively on R_3. In fact, due to the values taken by the resistance R_1, R_2 and R_3, we can write

$$gk = 1/R_1C_1 + 1/R_2C_1 + 1/R_3C_1 \approx 1/R_3C_1 \qquad (13)$$

and therefore only act on R_3 to vary the product *gK*.

With the ranges of variation chosen for the resistance coils, we can act on the system parameters as follows: *a=0.31¸1.39; b=0.68¸4.59; K=628¸8784; gK=48.4¸9469.*

Therefore, by acting on the organs controlling the parameters *a*, *b* and *g*, it is possible to change their characteristics in such a way as to observe the behaviour discussed above.

To visualise the effect of parametric variations on the shape of the potential *V(x)* a virtual tool was implemented using the LabVIEW® code, which gives the user a clear, efficient representation of the state of the system. The front panel of the tool is shown in Figure 5. As can be seen, the tool not only shows the evolution of the potential *V(x)* but also gives information about the values of the system parameters.

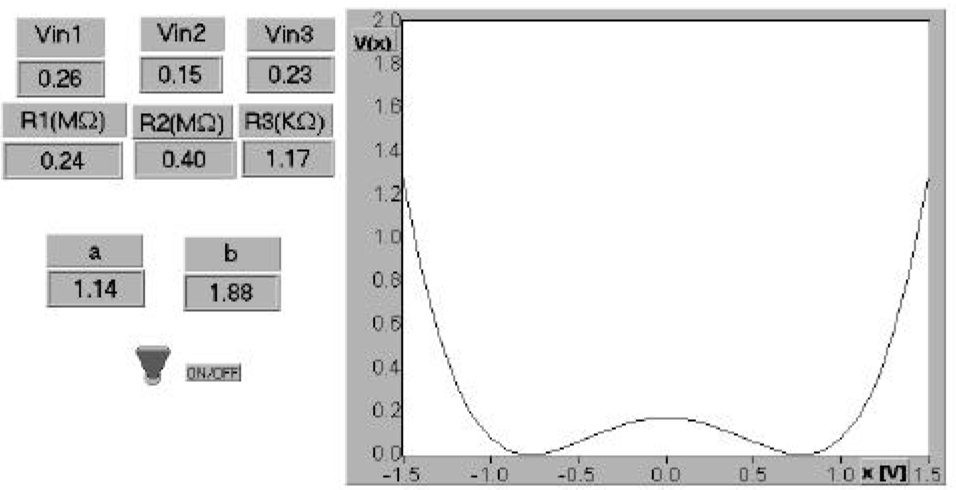

Figure 5. Front panel of the virtual instrument allowing for QDW system analysis.

The interface between the virtual tool and the analog device implementing the QDW is a data acquisition board (DAQ). The logical scheme on which the tool is based is shown in Figure 6.

One of the investigations performed to test the effectiveness of the structure, measured the values of the amplitude *A* and the frequency ν of the forcing signal needed to make the system switch, varying the inter-well distance.

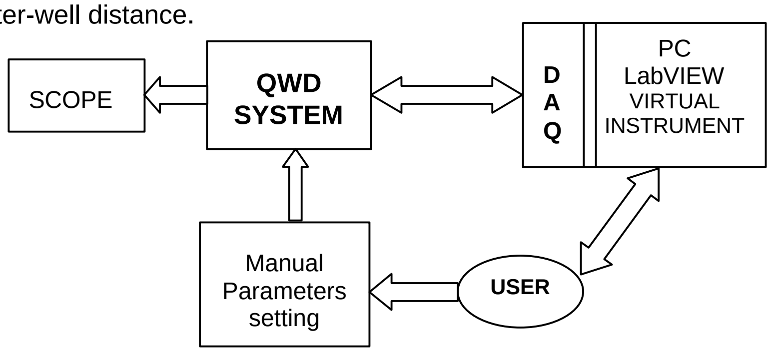

Figure 6. Logical scheme of the teaching tool.

The result of the analysis is summarised in Figure 7. The graphs give the optimal values for the distance between the two wells, in two conditions: a constant amplitude with a varying frequency and a constant frequency with a varying amplitude. As observed via simulation, to guarantee optimal system performance it is necessary, as the frequency increases, to reduce the distance between the two wells; likewise, as the amplitude *A* increases, the distance has to be increased.

Besides its usefulness as a technique for experimental investigation, the system is of great interest for teaching purposes. Observing on a screen of an oscilloscope how the state trajectories of a complex system like the QDW are modified when the shape of the potential changes (this can also be observed through a user interface) is of significant help in understanding such complex phenomena. Besides the fundamental support of the theoretical background and confirmation received via simulation, in fact, students or researchers studying complex phenomena frequently require confirmation via experimentation. In this sense the synergy between practice and theory, hardware and software, an analog implementation and a synoptic interface, provide a concrete completion of the process of cognitive acquisition of certain classes of phenomena. In this context, tools like the one discussed here play a fundamental role for understanding the behaviour of added-noise systems and their sensitivity to parametric variations.

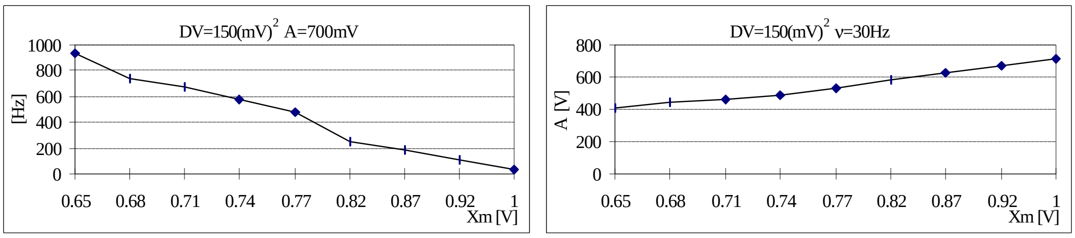

Figure 7. Inter-well distance at different values of both the forcing signal frequency and amplitude.

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