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UNCERTAINTY OF NON-UNIFORM QUANTIZATION AND INFORMATION

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Abstract – In many analog-to-digital conversion schemes, numerical values are attained with successive approximations of the difference between the reference and measured quantity. For effectiveness of differential tracking, the non-uniform quantization must fulfil three conditions: partitions into halves, increasing quantization uncertainty with difference, and low overlapping of the quantization intervals. The best trade between the number of decision levels and the settling time is with pure exponential quantization rule. The fastest response is achievable with base 2. The information rate per step is a little below three. The number of bits decreases towards the end of conversion and the sharp stop of the uniform constant information flow is smoothed.

Keywords: non-uniform quantization, uncertainty, information entropy

1. INTRODUCTION

Quantization, as part of the A/D (analog-to-digital) conversion, has great accent on the rate-distortion theory [1]. As the main support for optimization of the rate-distortion trade, the probability density function $f(g)$ of the signal $g(t)$ is used [2]. Digitalization consists of prefiltering, sampling, windowing, and quantization. Besides this, the dynamics of measurement A/D channel is also important. Here the trade between the number of references for generating the reference levels and the number of steps of the conversion is presented. Optimal results for the A/D conversion with regard to the time of conversion, resolution, and used references are obtained with the multi-step parallel technics [3].

Numerical value is attained with successive approximation of the difference between the reference quantity $g_r(t)$ and the measured quantity $g(t)$. The estimation of difference quantity has an error that is proportional to the difference with some constant added [4]. For effectiveness of differential tracking, the non-uniform quantization must fulfil three conditions: partitions into halves, increasing quantization uncertainty with difference, and low overlapping of the quantization intervals. The best trade between the number of decision levels and the settling time is with pure exponential quantization rule. The fastest response is achievable with base 2 [5] [6].

Quantization of a certain measurement interval gives a finite set of output levels or representatives *N*. At the initial state of the measurement approximation, the uncertainty of site of the searched signal is determined by the measurement range. At the end of the measurement procedure, the uncertainty interval is reduced to the uncertainty of the resolution interval - the smallest quantization interval Δ_0. Some information is gained in the Hartley sense $I = \log_2(N)$ [9], [10], if one representative among a certain number of alternatives is determined.

The uniform quantization equalizes the probabilities of individual representatives, assuming the constant probability density function of the searched signal in the measurement range $f(g) = 1/(g_{\max} - g_{\min})$. With the non-uniform quantization in the A/D conversion procedure, the probabilities p_j of representatives are no more equal, if $f(g)$ is constant. It is better to use the Shannon probabilistic definition of uncertainty - the Shannon entropy [11].

The finite impulse response of the A/D channel causes the finite information capacity of the channel. If settling time of the A/D conversion is cut, not all of possible values are approximated by the same minimal quantization uncertainty. From the information point of view, the entropy contributions of last steps can be negligible to the whole expected information.

2. PRELIMINARIES OF NON-UNIFORM QUANTIZATION

The exponential distribution function of representatives y_i with base $a = 2$ has several advantages: simplicity of the mathematical implement, exponential emphasis on the surroundings of origin, and possible practical realization [5], [6].

$$y_i = a^{i-1}\Delta_0 = 2^{i-1}\Delta_0 \qquad i = 1, 2, \ldots, n \qquad (1)$$

There must be symmetry around the origin y_0 (Fig. 1) to attain effectiveness of quantization (minimum distortion) and the fastest response of conversion [7], [8]. Index *i* of representatives obtain a negative sign, but their distance to y_0 remains the same $|y_i - y_0| = |y_{-i} - y_0|$.

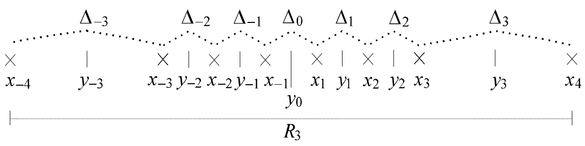

Fig. 1. Representatives and threshold levels of non-uniform quantization for $i = 3\,(a = 2)$

The first threshold - decision level x_1 must lie in the middle of y_1 and y_0 ($x_1 = \Delta_0/2$). This assures that the smallest quantization interval Δ_0 is that around the origin (($\Delta_0 = R/\left(2^n - 1\right)$; R - full scale of the measurement range; n - number of bits.). Other decision levels can be obtained by induction, if the generic function $x_{i+1} = 2y_i - x_i$, x_1, and y_1 are known. The expression for x_i ($a = 2$) can be written as:

$$\frac{x_i}{\Delta_0} = (-1)^i \left(2\frac{1-(-a)^{i-1}}{1-(-a)} - \frac{1}{2} \right) = (-1)^i \left(\frac{2+(-2)^i}{3} - \frac{1}{2} \right) \quad (2)$$

The distance of y_i from y_0 increases the uncertainty of the representative $\Delta_i/2$ or the quantization interval Δ_i. The quantization interval is fixed with the difference of adjacent thresholds.

$$\frac{\Delta_i}{\Delta_0} = \frac{\left(x_{i+1} - x_i\right)}{\Delta_0} = \frac{2a^{i-1}(a-1)}{1+a} + (-1)^{i+1}\frac{3-a}{1+a}$$

$$\frac{\Delta_i}{\Delta_0} = \frac{1}{3}\left(2^i + (-1)^{i+1}\right) \quad (3)$$

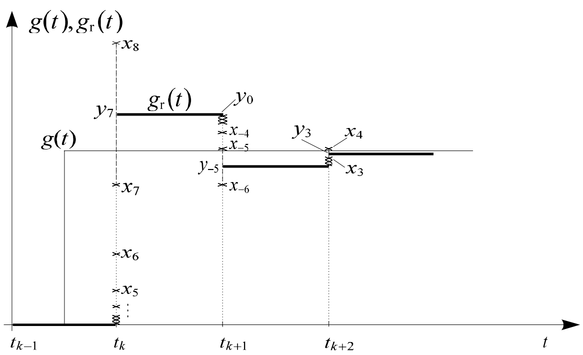

Fig. 2. Approximation procedure of non-uniform error reduction between signal g and reference quantity g_r

The speed of the approximation procedure for different structures of A/D conversions is measured by the number of steps to achieve the basic resolution around the searched value (Fig. 2).

The number of steps of the A/D conversion with exponential differential quantization depends on the signal step at the input. The larger as the signal change is, the larger is the number of approximation steps to within $\pm\Delta_0/2$. Problems of the worst case settling time are similar to the problems of the range's width (measured signal's step), which can be examined at the fixed number of approximation steps (Fig. 3). For the base $a = 2$, two ways of expansion of the examination interval are possible, because there are two possibilities in reduction within the smallest quantization uncertainty $\pm\Delta_0/2$ in the last step ($k_{\max-1} \to k_{\max}$): y_1 ($y_1 + \Delta_1/2 \to \Delta_0/2$) and y_2 ($y_2 + \Delta_2/2 \to \Delta_0/2$).

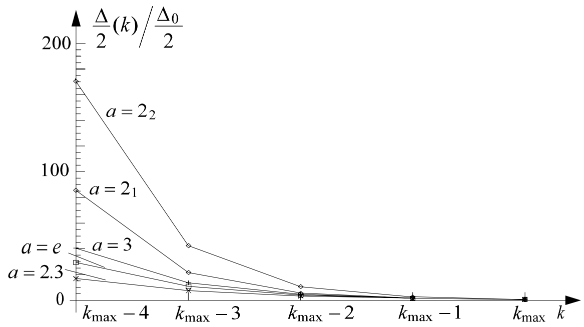

Fig. 3. Uncertainty of the worst case settling to $\pm\Delta_0/2$ during the last four steps ($a = 2_1, 2_2, 2.3, e, 3$)

3. INFORMATION FLOW

The principle that the larger as the distance is, the larger is also the dynamic error of estimation shows the limitation of the information flow. Some information is gained, if one representative among a certain number of alternatives is determined. It comes out that the information flow is not constant during approach to the searched value, since the number of steps differs with the signal position.

3.1. Probability distribution of the number of steps

In the worst case of approaching to the searched value, the index is reduced only for two. There are also positions of the searched signal to which the approximation steps y_i approach within the borders of $\pm\Delta_0/2$ and the procedure of approaching is finished in one step. The question is: what is the probability of appearance of the approaching path with a given number of steps k ($k_0 = 0$, $k_1 = 1$, ..., $k_{\max}$)?

The probability density function of the measured signal $f(g)$ is assumed constant in the measurement range like in many A/D conversion technics. Increasing of index i increases the uncertainty interval of the first step - the range of examination $\left(d_i = y_i + \Delta_i/2;\ i = 0, 1, 2, \ldots, n\right)$ (Fig. 4).

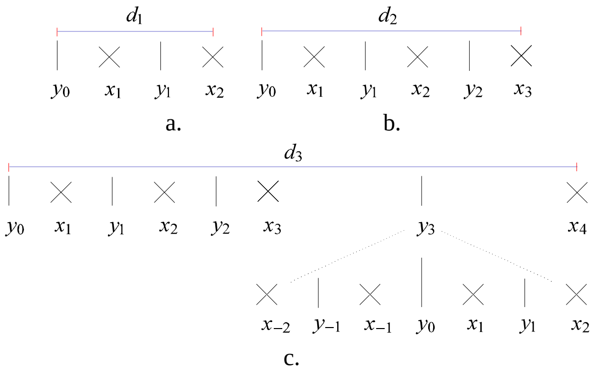

Fig. 4. Ranges of examination for the first three values of index i

In these intervals, the probability density functions are constant $f_i(g) = 1/d_i$ ($0 \le g \le d_i$). For the base $a = 2$ the following relations are valid ($d_i = \left(2^{i+2} + (-1)^{i+1}\right)\Delta_0/6$).

- $i = 0$: $d_0 = y_0 + \Delta_0/2 = \Delta_0/2$ ... in this trivial case, the signal is positioned within $\Delta_0/2$ and the number of steps is zero.
- $i = 1$ (Fig. 4a): $d_1 = y_1 + \Delta_1/2 = 3 \cdot \Delta_0/2$

$$\bar{k} = p_0 k_0 + p_1 k_1 = \int_{y_0}^{x_1} f_1(g)\mathrm{d}g \cdot k_0 + \int_{x_1}^{x_2} f_1(g)\mathrm{d}g \cdot k_1 =$$
$$= f_1(g)\left(\frac{\Delta_0}{2} \cdot k_0 + 2\frac{\Delta_0}{2} \cdot k_1\right) = \frac{\Delta_0}{2d_1}(k_0 + 2k_1) =$$
$$= \frac{1}{3}(k_0 + 2k_1) = 0{,}66 \qquad (4)$$

The probability p_1 of path with one step is $2/3$ and without step p_0 is $1/3$. The examination is finished on the uncertainty of $\pm\Delta_0/2$ and the average number of steps is 0.66.

- $i = 2$ (Fig. 4b): $d_2 = y_2 + \Delta_2/2 = 5 \cdot \Delta_0/2$

$$\bar{k} = \int_{y_0}^{x_1} f_2(g)\mathrm{d}g \cdot k_0 + \int_{x_1}^{x_2} f_2(g)\mathrm{d}g \cdot k_1 + \int_{x_2}^{x_3} f_2(g)\mathrm{d}g \cdot k_1 =$$
$$= \frac{1}{5}(k_0 + 4k_1) = 0{,}8 \qquad (5)$$

In the probability of the path with one step $p_1 = 4/5$, there participate two equal length quantization intervals $\Delta_1 = x_2 - x_1$ and $\Delta_2 = x_3 - x_2$.

- $i = 3$ (Fig. 4c): $d_3 = y_3 + \Delta_3/2 = 11 \cdot \Delta_0/2$

$$\bar{k} = \int_{y_0}^{x_1} f_3(g)\mathrm{d}g \cdot k_0 + \int_{x_1}^{x_3} f_3(g)\mathrm{d}g \cdot k_1 +$$
$$+ 2\left(\int_{y_0}^{x_1} f_3(g)\mathrm{d}g \cdot k_0 + \int_{x_1}^{x_2} f_3(g)\mathrm{d}g \cdot k_1\right) \cdot k_1 =$$
$$= \frac{\Delta_0}{2d_3}\left(k_0 + 4k_1 + 2(k_0 + 2k_1)k_1\right) = \frac{1}{11}\left(k_0 + 4k_1 + 2k_{1+0} + 4k_{1+1}\right)$$
$$\bar{k} = \left(\frac{1}{11}k_0 + \frac{6}{11}k_1 + \frac{4}{11}k_2\right) = 1{,}2727 \qquad (6)$$

If y_3 is the approximation to the signal in the first step, than one more step is possible to achieve $\pm\Delta_0/2$. The term $k_{1+0} = k_1 = 1$ denotes the path with one step to all values in the interval $y_3 - \Delta_0/2 < g \le y_3 + \Delta_0/2$. The term $k_{1+1} = k_2 = 2$ is used for the path with two steps ($\rightarrow y_3 \rightarrow y_{+1,-1}$). The double probability at y_3 is obtained, because the approximation y_3 in the first step becomes the origin y_0 in the next step with symmetrically arranged thresholds x_i and x_{-i} around.

With increasing of the value *i*, the average number of steps $\bar{k}$ can be written by the induction and the way of determination the path probability of the k steps ($k_{1+2} = k_3, \ldots$) should be considered.

Example $i = 16$:

$$\bar{k} = \left(\frac{1}{87381}k_0 + 0{,}0004k_1 + 0{,}0048k_2 + 0{,}0333k_3 + 0{,}1309k_4\right.$$
$$\left. + 0{,}2900k_5 + 0{,}3384k_6 + 0{,}1758k_7 + 0{,}0264k_8\right) = 5{,}55 \quad (7)$$

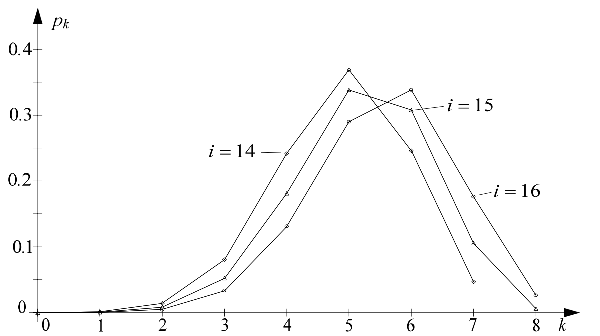

Fig. 5. Probability distributions of steps at $i = 14, 15, 16$

The approximation of an arbitrary value with the quantization uncertainty of $\pm\Delta_0/2$ has the probability distribution of steps like in Fig. 5. The average number of steps limits to two thirds of the maximal value.

3.2 Information content

With the non-uniform quantization in the A/D conversion procedure, the probabilities p_j of representatives are no more equal, if $f(g)$ is constant. It is better to use the Shannon probabilistic definition of uncertainty - the Shannon entropy [11].

$$H = -\sum_j p_j \log_2 p_j \qquad (8)$$

The measurement range R, calibrated with the resolution interval, is the whole interval around the origin y_0 within the borders $\pm(y_i + \Delta_i/2)$ (Fig. 1).

$$R_i = 2d_i = 2y_i + \Delta_i = \left(a^{i-1}\frac{4a}{a+1} + (-1)^{i+1}\frac{3-a}{a+1}\right)\Delta_0 \qquad (9)$$

If the range remains unchanged, the smallest quantization interval relatively decreases by increasing of i. The probability of representative's occurrence is proportional to the width of the belonging quantization interval.

$$p_j = \frac{\Delta_j}{R_i}; \qquad j = -i, \ldots, -1, 0, 1, \ldots, i \tag{10}$$

Now the entropy for the range R_i can be written:

$$H_i = -\left(p_0 \log_2 p_0 + 2\sum_{j=1}^{i} p_j \log_2 p_j \right). \tag{11}$$

The entropy can be expressed by the quantization intervals as follows.

$$H_i = -\frac{\Delta_0}{R_i}\left(2\sum_{j=1}^{i}\frac{\Delta_j}{\Delta_0}\log_2\left(\frac{\Delta_j}{\Delta_0}\right) - \log_2\left(\frac{R_i}{\Delta_0}\right)\left(1 + 2\sum_{j=1}^{i}\frac{\Delta_j}{\Delta_0}\right)\right) \tag{12}$$

In (12) the range and the quantization interval are normalized by the minimal quantization interval.

From (12), (9), and (3) can be deduced, that increasing of the i or of the number of representatives does not increase the entropy proportionally $H_{\max} = \log_2(R_i/\Delta_0)$. It limits to the constant value ($H(a=2) \underset{i\to\infty}{=} 3{,}0000$ bits, $H(a=3) \underset{i\to\infty}{=} 2.3774$ bits).

The proposed non-uniform exponential quantization with its limited information rate shows one of the possible ways, how a measurement system with a limited capacity of channel ($c = H/\text{step} = \text{const.}$) can be adapted to the increased information rate at the input ($i \to \infty$).

3.3 Expected rate of information in the successive steps

The expected information of the first step is determined by the probabilities of possible representatives, which could become the origins in the next step of examination. The information content is expressed as the weighted arithmetic mean of uncertainties $-\log_2 p_j$, which is exactly the Shannon entropy. Because the quantization intervals have no overlapping, and the probability density function of the searched signal is constant in the measurement range, the probabilities are proportional to the widths of quantization intervals.

During the next steps, the branching property of the Shannon entropy is used. In the second step, the entropy depends upon probabilities of the paths, which have remained for examination to the final approximation within $\pm\Delta_0/2$, and upon the entropy, that is obtained during the first steps of continuation on these paths.

As an illustration, the example of $i = 3$, $a = 2$ (Fig. 1) is analysed. According to (12), the entropy of the first step is ${}_1H = H_i = H_3 = 2{,}595$ bits. In the second step, the intervals around representatives y_{-3} and y_3 ($\Delta_{-3}, \Delta_3 > \Delta_0$) have remained for examination. They have equal probabilities $p_3 = \Delta_3 / R_3$. The information, which is obtained during one of these intervals, is equal $H_i = H_1$ (12).

The entropy for the second step is ${}_2H = 2p_3H_1 = 2\cdot\Delta_3/R_3\cdot H_1 = 2\cdot 3/11\cdot 1{,}585 = 0{,}8645$ bits. The sum of entropies of both steps gives us the maximal information for the measurement range $R_3 = 11\Delta_0$ with the resolution of Δ_0.

$$H(i=3) = {}_1H + {}_2H = \log_2(11) = 3{,}4594 \text{ bits} \tag{13}$$

Similar considerations are also valid for larger values of *i*. The number of steps $k_{\max}$ is increased for 1 at odd values of the index ($k_{\max}(i=1) = 1$, $k_{\max}(i=3) = 2$, $k_{\max}(i=5) = 3$, ...).

The entropy of the first step of approximation is expressed by (12).

1. step: ${}_1H = H_i$ (14)

For the second step, the reflection from example can be generalized:

2.step: ${}_2H = 2(p_3H_1 + \ldots + p_iH_{i-2}) = 2\frac{1}{R_i}\sum_{j=3}^{i}\Delta_j H_{j-2}$ (15)

During the remaining steps of the A/D conversion the index *i* is reduced by two. The representatives with the index above $i = 5$ enable following paths and their entropies:

$$y_5 \to 2p_3H_1$$

$$y_i \to 2(p_3H_1 + p_4H_2 + \cdots + p_{i-2}H_{i-4}) \tag{16}$$

The decrease of constants a_j, besides entropy contributions p_jH_{j-2} in steps *k* for a given *i*, can be written in the form of series:

$${}_{k=2}a_j = 1$$

$${}_3a_j = i - j - 1 \qquad j = 3, 4, \ldots, i-2$$

$${}_4a_j = (i - j - 3)\left(\frac{(i-j-3)+1}{2}\right) \qquad j = 3, 4, \ldots, i-4$$

$${}_ka_j = \frac{1}{(k-2)!}\prod_{m=0}^{k-3}(i - j - 2k + 5 + m) \qquad j = 3, 4, \ldots, i - 2(k-2) \tag{17}$$

The entropy for the third step can be generally obtained with induction.

3. step: ${}_3H = 4\frac{1}{R_i}\sum_{j=3}^{i-2}(i - j - 1)\Delta_j H_{j-2}$ (18)

For the remaining higher steps, the entropy can be written as follows:

k.step:

$$ {}_kH = \frac{2^{k-1}}{R_i}\sum_{j=3}^{i-2(k-2)}\left(\frac{1}{(k-2)!}\prod_{m=0}^{k-3}(i-2k+5-j+m)\right)\Delta_j H_{j-2} \qquad (19)$$

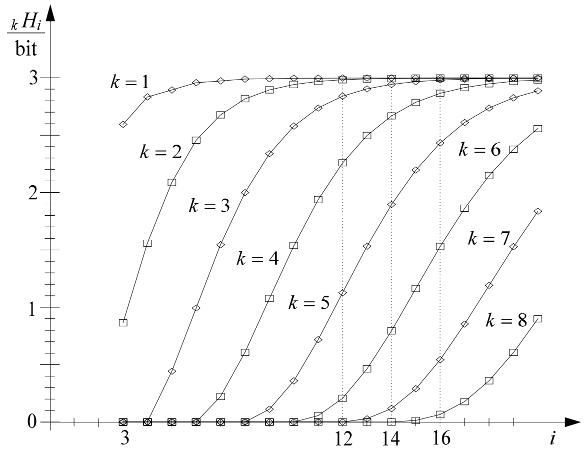

Fig. 6. Increasing of entropies in steps ($k = 1,\ldots,8$)

The entropy's curve of the *k*-th step in dependence of *i* monotonously increases from zero to three (Fig. 6). In (19) the expressions for Δ_j (3), R_i (9), and H_{j-2} (12) is used.

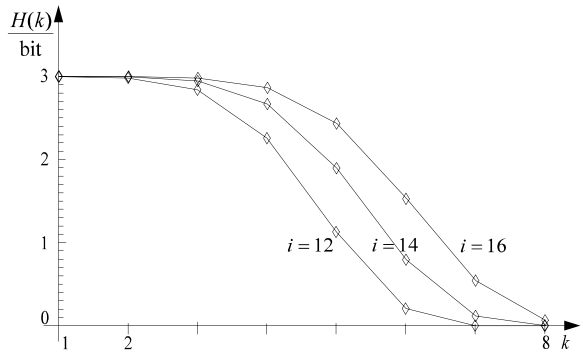

Fig. 7. Information rate per step during the conversion procedure ($i = 12,14,16$)

If we now collect the entropy's contribution of several steps at the same index *i* (Fig. 6 - vertical doted lines), it comes out that the information flow is not constant like in the uniform parallel-serial A/D conversions. The number of bits decreases towards the end of conversion and the sharp stop of the constant information flow is smoothed (Fig. 7).

At the beginning of the conversion procedure, the information rate per step is a little below three. During the last two steps it decreases. In dependence upon how much information is neglected, the last steps can be omitted. However, the sum of entropy's contribution of all steps is equal to the maximal information contents $H_{\max} = \log_2(R_i/\Delta_0)$ of the measurement range calibrated by Δ_0.

3. CONCLUSION

The principle that the larger as the distance is, the larger is also the dynamic error of estimation shows the limitation of the information flow.

In comparison with the uniform quantization, the information of the first step does not increase with increasing the number of bits n. It limits to the constant value $H(a=2)_{n\to\infty} = 3$.

The quantity of information decreases by approaching to the last step of conversion.

If settling time of the A/D conversion is cut, not all of possible values are approximated by the same minimal quantization uncertainty ($\pm\Delta_i/2 \neq \pm\Delta_0/2$). From the information point of view, the entropy contributions of last steps can be negligible to the whole expected information. The problem of measurement is 'one of all' and the problem of information is 'most of all'.

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