Uncertainty Evaluation: Analysis of Simultaneous Measurement of Multiple Measurands

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Keywords: Uncertainty, Confidence Region

ABSTRACT

The Guide to the expression of uncertainty in measurement (GUM) (ISO,1995) [1], contains internationally agreed recommendations for the evaluation and expression of uncertainties in a measurement process, providing valuable advise for constituting the model that relates input to output quantities.

If a probability density function is associated with each of input quantities, a probability "region" for the outputs quantities (the measurands) can be derived and its level of confidence may be evaluated by analytical or numerical methods.

This paper analyzes the uncertainty evaluation of a process concerning simultaneous measurement of multiple measurands on the base of the fundamental properties of the probability confidence region.

1. INTRODUCTION

The case of interest in this research is represented by a model concerning a number n ($n \geq 2$) of measurands determined simultaneously in the same measurement process.

To the output referred to the i-th measurand we associate the random variable M_i usually called i-th measure. We assume that the variability of the n-dimensional random vector $\underline{M} = (M_1,...,M_n)$, called measure vector, may be summarised into a probability region, say $C^{(n)} \subset R^n$, with a convenient level of confidence equal to p, so that $\mathrm{P}\{\underline{M} \in C^{(n)}\} = p$.

Introducing the n-dimensional joint probability density function: $f_{\underline{M}}(\underline{m})$, being $\underline{m} = (m_1, m_2, ... m_n)$ a generic realisation of the random vector $\underline{M} = (M_1,...,M_n)$, it is possible to establish the following identity relation:

$$\mathrm{P}\{\underline{M} \in C^{(n)}\} = \int \cdot \cdot \int_{C^n} f_{\underline{M}}(\underline{m})\; dm_1 \cdot \cdot dm_n = p \quad (1)$$

with :

$$f_{\underline{M}}(\underline{m}) = \lim_{\underline{h} \to \underline{0}} \frac{\mathrm{P}\{\underline{m} < \underline{M} \leq \underline{m} + \underline{h}\}}{\underline{h}} = \frac{\partial^n F(\underline{m})}{\partial m_1 \partial m_2 ... \partial m_n} \quad (2)$$

where $\underline{h} = (h_1, h_2, ...h_n)$ and

$F(\underline{m}) = \int_{-\infty}^{m_1}\int_{-\infty}^{m_2}...\int_{-\infty}^{m_n} f_M(x_1,...x_n)\,dx_1...dx_n$ is the probability distribution function.

It is important to underline that the specification of the content of a probability region $C^{(n)}$ does not uniquely define its frontier. To achieve uniqueness one must impose some constraints, for instance one could require that it should have minimal measure (length, area, volume, and so on), or that it should be "central", that is should have symmetric properties like a convex symmetric domain around the origin of the coordinate axes system.

It is possible to assess that any point in the probability region $C^{(n)}$ is representative of the n measurands. This region may be said, colloquially, to encompass a large fraction of the distribution of the measure-realizations in the sense that, in the long run, a large proportion of points attributed to the n measurands would lie in $C^{(n)}$. This fraction may be viewed as the coverage probability (or level of confidence) of the stated region.

The intuitive content of the concept is that, if the coverage probability is sufficiently large, we are practically certain that any realisation of the vector measure will lie in the probability region.

2. MODELS AND RESULTS

A theoretical n-dimensional model is considered and computational results concerning a uniform distribution are presented for a three-dimensional model.

3. THE N-DIMENSIONAL MODEL

Starting from the identity relations (1) and (2), the following general considerations may be introduced for a n-dimensional model:

- The relation that links the joint n-variate distribution function to the marginal ones, in the hypothesis that the correlation is significant only between two by two measures, is:

$$F_{\underline{M}}(\underline{m}) = \prod_{i=1}^{n} F_{M_i}(m_i) \bullet \left\{ 1 + \sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \theta_{rs}\left[1-F_{M_r}(m_r)\right]\left[1-F_{M_s}(m_s)\right] \right\} \quad (3)$$

where θ_{rs} $r,s = 1,...n$ is a parameter depending on the correlation coefficient ρ_{rs} and where $F_{M_r}(m_r)$ and $F_{M_s}(m_s)$ are the r-th and s-th respectively, marginal univariate distribution functions associated to the joint n-variate distribution function $F_{\underline{M}}(\underline{m})$.

- The n-th order partial derivative with respect $m_1, m_2, ...m_n$ of equation (3) gives the relationship linking the joint n-variate density function $f_{\underline{M}}(\underline{m})$, with the corresponding univariate marginal density functions $f_{M_j}(m_j)$, $j = 1,...n$, and the univariate distribution functions $F_{M_j}(m_j)$, $j = 1,...n$, that is:

$$f_{\underline{M}}(\underline{m}) = \prod_{i=1}^{n} f_{M_i}(m_i) \left\{ 1 + \sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \theta_{rs}\left[1-2F_{M_r}(m_r)\right]\left[1-2F_{M_s}(m_s)\right] \right\} \quad (4)$$

- The link between θ_{rs} and ρ_{rs} follows from the equalities:

$$\mathrm{Cov}\{M_r, M_s\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} m_r m_s f_{M_r M_s}(m_r, m_s)\, dm_r\, dm_s - E\{M_r\}E\{M_s\} = \rho_{rs}\sqrt{\mathrm{Var}\{M_r\}\mathrm{Var}\{M_s\}} \quad (5)$$

with $E\{M_i\} = \int_{a_i}^{b_i} m_i\, f_{M_i}(m_i)\, d\, m_i$ $i = 1,2$ and

$\mathrm{Var}\{M_i\} = \int_{a_i}^{b_i} [m_i - E\{M_i\}]^2 f_{M_i}(m_i)\, d\, m_i$

where $Var\{M_r\}$, $Var\{M_s\}$ and $Cov\{M_r, M_s\}$ are, respectively, the variances and the covariance of M_r, M_s.

From (4) and (5), taking into account the equality:

$$\int_{-\infty}^{+\infty} f_{M_i}(m_i)\,[1-2F_{M_i}(m_i)]dm_i = 0 \quad i=1,2,...n$$

we obtain for each couple $(r,s): r=1,...,n-1, s=r+1,...,n$:

$$\theta_{rs} = \frac{\rho_{rs}\sqrt{Var\{M_r\}Var\{M_s\}}}{K} \quad \text{with:}$$

$$K = \prod_{i=r,s}\left\{\int_{-\infty}^{+\infty} m_i f_{M_i}(m_i)\left[1-2F_{M_i}(m_i)\right]dm_i\right\} \qquad (6)$$

Computationally, a three-dimensional model has been considered and the results concerning a uniform three-variate distribution are presented.

3.1 A three-dimensional model

If we assume a convex domain $C^{(3)} \subset R^3$ as the probability region with level of confidence equal to p, we have $C^{(3)} \subseteq I_1 \times I_2 \times I_3$ where $I_j = \lfloor a_j, b_j \rfloor$, $j=1,2,3$, are the projection of $C^{(3)}$ on the co-ordinate axes.

The computational results to get an estimation of the parameter p may be reached by a numerical approximation of the multiple integral:

$$\iiint_{C^{(3)}} f_{\underline{M}}(\underline{m})dm_1 dm_2 dm_3 = p \qquad (7)$$

with $\underline{m} = (m_1, m_2, m_3)$ and $\underline{M} = (M_1 M_2 M_3)$ according to the probability distribution function $f_{\underline{M}}(\underline{m})$ representing the model.

In particular, if $C^{(3)} \equiv I_1 \times I_2 \times I_3$, $I_j = \lfloor a_j, b_j \rfloor$, $j=1,2,3$, the level of confidence p is equal to:

$$F_{\underline{M}}(b_1,b_2,b_3) + F_{\underline{M}}(a_1,a_2,b_3) +$$
$$+F_{\underline{M}}(b_1,a_2,a_3) + F_{\underline{M}}(a_1,b_2,a_3) -$$
$$-F_{\underline{M}}(b_1,b_2,a_3) + F_{\underline{M}}(b_1,a_2,b_3) +$$
$$+F_{\underline{M}}(a_1,b_2,b_3) + F_{\underline{M}}(a_1,a_2,a_3) = p \qquad (8)$$

The three random variables M_1, M_2, M_3, representing the measurement results, are supposed to have a uniform distribution, that is each marginal probability density function has the form:

$$f_{M_i}(m_i) = \begin{cases} \dfrac{1}{d_i - c_i} & if\ c_i \le m_i \le d_i \quad i=1,2,3 \\ 0 & otherwise \end{cases} \qquad (9)$$

and the correspondent distribution function is:

$$F_{M_i}(m_i) = \begin{cases} 0 & if\ m_i < c_i \quad i=1,2,3 \\ \dfrac{m_i - c_i}{d_i - c_i} & if\ c_i \le m_i \le d_i \quad i=1,2,3 \\ 1 & if\ m_i > d_i \quad i=1,2,3 \end{cases} \qquad (10)$$

Taking into account that :

$$\mu_i = E\{M_i\} = \frac{c_i + d_i}{2} \quad ,$$

$$\sigma_i^2 = Var\{M_i\} = \frac{(d_i - c_i)^2}{12} \quad i=1,2,3$$

and thus :

$$d_i - c_i = 2\sqrt{3Var\{M_i\}} \quad i=1,2,3$$

we have:

$$\int_{c_i}^{d_i} m_i f_{M_i i}(m_i)\left[1-2F_{M_i}(m_i)\right]dm_i =$$

$$-\frac{2Var\{M_i\}}{d_i - c_i} = -\sqrt{\frac{Var\{M_i\}}{3}}, i=1,2,3$$

and, consequently, from (6), since $K = \dfrac{\sqrt{Var\{M_r\}Var\{M_s\}}}{3}$, we obtain $\theta_{rs} = 3\rho_{rs} \quad r \ne s = 1,2,3$.

The confidence belt within a stated level of confidence p, may be evaluate by the equalities (4) and (8). In the following tables are reported, for different values of $\rho_{rs}, r \neq s = 1,2,3$, the computational results of the coverage probability p for the confidence belt corresponding to the following values of a_i and b_i:

$$a_i = \mu_i - k\sigma_i, \quad b_i = \mu_i + k\sigma_i \quad \mathrm{i} = 1,2,3$$

with $\mu_i = \dfrac{d_i + c_i}{n_i}$, $i = 1,2,3$, and for different values of k, and being $\sigma_i = \dfrac{|d_i - c_i|}{2\sqrt{3}}, \mathrm{i} = 1,2,3$

Tab.1 The confidence level p-values for $\rho_{rs} = 0.5, \quad \mathrm{r} \neq \mathrm{s} = 1,2,3$; $n_i = 3 \ \mathrm{i} = 1,2,3$; $c_i = -1, d_i = 2 \quad \mathrm{i} = 1,2,3$

k	1	1.5	1.7
p	0.1925	0.6495	0.9455

6. CONCLUSIONS

The uncertainty evaluation of a process concerning simultaneous measurement of multiple measurands has been analised on the base of the probability confidence region.

Numerical results have been presented for a three-dimensional model with uniform distribution.

The analysis may be extended to more general models.

7. REFERENCES

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