Uncertainty’s Analysis of Impedance Measurements In the Sampling Sensor Instrument

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ABSTRACT

The algorithms based on digital signal processing for impedance components’ evaluation in circuits with sampling transducer have been analysed. It is supposed that the voltage and current are sampled synchronously to the fundamental frequency of the generated sinusoidal signal. Two fitting sine wave algorithms, which are based on the least mean square (LMS) technique, have been described. The first one reconstructs indirect measurement method. The second algorithm estimates the unknown impedance components by direct method. The uncertainty’s propagation by described algorithms can be analysed by means of covariance matrix. The elements of covariance matrix are dependent on the phase angles of the measurement signals. It is shown that those algorithms provide minimization of uncertainty for selected number of samples and phase angles. The influence’s simulations of quantization error of the AD converter and jitters of sampling time for described algorithms on uncertainty processing results have been carried out.

Keywords: Sampling Sensor, Impedance Measurements, LMS Method, Fitting Sine Wave Algorithms, Covariance Matrix, and Uncertainty Analysis.

1. INTRODUCTION

For many years, a lot of papers [1]-[5] have presented various types of circuits, techniques, and instruments for impedance measurement. In many circuits high accuracy is obtained using sampling transducer and digital algorithm for measuring unknown impedance. In these circuits the components of measuring impedance are calculated on the grounds of values of voltage drop *u*(*t*) and current *i(t)* flowing into the unknown impedance Z_x, which are simultaneously sampled by the dual channel data acquisition system. Some of used algorithms are based on the least mean square (LMS) method [1], [4].
The hardware requirements for proposed measurement technique include:

- A sine-wave generator to generate, with known frequency f_g, a voltage *u(t)* on the unknown impedance Z_x,
- A voltage-to-current converter to measure a current *i(t)*,
- And a dual channel acquisition system for simultaneous sampling of those signals, with frequency f_s, synchronously to frequency f_g.

In this paper the propagation of uncertainty throughout the algorithm is studied in some detail. The key parameters that affect the performance of the algorithm for short records of samples are derived.

2. LEAST SQUARE FIT TO SINE WAVE ALGORITHMS

Both sampled signals *u(t)* and *i(t)* can be written in the time domain as

$$u(t) = U_m \sin(wt + y_u), \quad i(t) = I_m \sin(wt + y_i), \tag{1}$$

where U_m, R_u, and I_m, R_i indicate the unknown amplitude and phase angle of first and second signal, respectively. It is supposed that signals described in Eq. (1) are sampled with sampling frequency f_s synchronously to the fundamental frequency f_g of the generated sinusoidal signal. Assume that the data record {*u(n)*} and {*i(n)*} contain the sequence of *N* samples for each signal, taken at time instants nT_s (n=0,1,...,N-1). It is further assumed that both signals can be modelled by

$$y(n) = X_c \sin wnT_s + X_s \cos wnT_s, \tag{2}$$

where X_c and X_s are unknown constants [6].
Expression (2) can be written in the matrix form as follows

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sin wT_s & \cos wT_s \\ \vdots & \vdots \\ \sin wnTs & \cos wnT_s \\ \vdots & \vdots \\ \sin w(N-1)T_s & \cos w(N-1)T_s \end{bmatrix} \begin{bmatrix} X_c \\ X_s \end{bmatrix} = \mathbf{AX}. \tag{3}$$

Estimates of the unknown parameters in **X** are obtained by a least mean squares method. Eq. (3) is an overdetermined (if N>2) set of linear equations, with the LMS solution given by

$$\hat{\mathbf{X}} = (\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{y}. \tag{4}$$

Having estimated vectors of orthogonal components of voltage and current, then the components of impedance can be obtained as

$$\hat{R}_x = \frac{\hat{U}_c \hat{I}_c + \hat{U}_s \hat{I}_s}{\hat{I}_c^2 + \hat{I}_s^2}, \quad \hat{X}_x = \frac{\hat{U}_s \hat{I}_c - \hat{U}_c \hat{I}_s}{\hat{I}_c^2 + \hat{I}_s^2}, \tag{5}$$

where $\hat{U}_c$, $\hat{I}_c$ and $\hat{U}_s$, $\hat{I}_s$ indicate the estimates of real and imaginary components of voltage and current, respectively. Eq. (5) is suitable for series equivalent of the unknown impedance Z_x. The number N of recorded samples per channel is connected to the number M of samples per period T_g by

$$T_m = NT_s = \frac{N}{M} T_g \, , \tag{6}$$

where T_m indicates the acquisition time, N and M are natural numbers. If N/M is integer, then $(\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}$ is a 2×2 diagonal matrix, and Eq. (4) can be written as

$$\hat{\mathbf{X}} = \frac{2}{N} \mathbf{A}^{\mathbf{T}} \mathbf{y} = \frac{2}{N} \left[\mathrm{Im}(\hat{Y}) \;\; \mathrm{Re}(\hat{Y}) \right]^{\mathbf{T}} . \tag{7}$$

The elements of vector $\hat{\mathbf{X}}$ are complex components of DFT of data record $\mathbf{y}$.

It is possible to start the sampling process with other initial conditions than showed by Eq. (3). For arbitrary fixed initial phase angle a a new input matrix $\mathbf{A}_{\alpha}$ can be written as follows

$$\mathbf{A}_a = \mathbf{A}\mathbf{T}^{\mathbf{T}} \tag{8}$$

where

$$\mathbf{T} = \begin{bmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{bmatrix} \tag{9}$$

is the transformation matrix [7].

However the LMS solution of the unknown parameters in $\mathbf{X}$ is still obtained by Eq. (3), putting $\mathbf{A}=\mathbf{A}_{\alpha}$.

Algorithm described above reconstructs indirect measurement method. In order to appoint compound standard uncertainty the law of error propagation is used.

On the other hand the voltage $u(t)$ and the current $i(t)$ flowing into impedance Z_x can be described in terms of equation as follows

$$u(n) = |Z_x| I_m \sin(wnT_s + y_i + j) \, , \tag{10}$$

where $|Z_x| = \sqrt{R_x^2 + X_x^2}$, $j = \arctan \frac{X_x}{R_x} = y_u - y_i$.

Expression (10) can be expanded similar to Eq. (2) as

$$u(n) = R_x I_m \sin(wnT_s + y_i) + X_x I_m \cos(wnT_s + y_i) \tag{11}$$

Considering that the sequence of current samples $\{i(n)\}$ satisfies Eq. (1), Eq. (11) can be written as

$$u(n) = R_x i(n) + X_x i\left(n + \frac{M}{4}\right) . \tag{12}$$

In this case, to create the matrix **A,** the sequence of current samples $\{i(n)\}$ and the sequence shifted by $M/4$ corresponding to it, can be used. Values of impedance components can be calculated directly from Eq. (4).

3. UNCERTAINTY'S EVALUATION IN IMPEDANCE COMPONENTS MEASUREMENTS

To obtain uncertainty of result of impedance components measurement it is necessary to use the law of uncertainty propagation [7], [8]. Estimation of uncertainty value needs evaluation of the elements of measurand's covariance matrix [7]-[9].

The input matrixes **A** for indirect method are determinate. Assuming, that errors $\boldsymbol{\varepsilon}_{\mathbf{u}}$ and $\boldsymbol{\varepsilon}_{\mathbf{i}}$ in data are zero-mean noises with variances s_u, s_i, covariance matrixes of voltage and current components are given by formula

$$\mathbf{C}_y = \mathrm{cov}(\mathbf{X}_y) = s_y^2 (\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}, \quad y = u, \; i \, . \tag{13}$$

If the signals are modelled by Eq. (1), the covariance matrix can be transformed to form

$$\mathbf{C}_y = \frac{s_y^2}{\det(\mathbf{A}^{\mathbf{T}}\mathbf{A})} \begin{bmatrix} \sum_{n=0}^{N-1} \cos^2 wnT_s & -\sum_{n=0}^{N-1} \sin wnT_s \cos wnT_s \\ -\sum_{n=0}^{N-1} \sin wnT_s \cos wnT_s & \sum_{n=0}^{N-1} \sin^2 wnT_s \end{bmatrix} \tag{14}$$

Finally, after summation of the components, $\mathbf{C}_{\mathbf{y}}$ can be written as Eq. (15), shown at the bottom of the page. One can note that variances and covariances of voltage and current components are dependent on the acquisition time length T_m. Only if $T_m = kT_g/2$ $(k \in \mathcal{N})$, the extradiagonal elements of covariance matrix are equal to zero and the components of voltage U_c, U_s and current I_c, I_s can be independently estimated.

If the input matrixes are described by Eq. (8), the covariance matrix can be written as [7]

$$\mathbf{C}_{y,a} = \mathbf{T}\mathbf{C}_y\mathbf{T}^{\mathbf{T}} \qquad y = u, \; i \, . \tag{16}$$

Therefore the elements of covariance matrixes $\mathbf{C}_{y,\alpha}$ depend on the selected value of initial phase angle a.

In case of direct method, the elements of input matrixes **A** are results of current measurements. That is why we cannot assume that measuring errors are statistically independent. To estimate impedance components it is necessary to use the method of generalized least squares [9]. As the $N \times N$ errors matrix is unknown, for the purpose of an approximate analysis one can use the covariance matrix described by Eq. (15). If matrix **A** is determinate, the diagonal elements of covariance matrix $\mathbf{C}_{\mathbf{z}}$ are a measure of uncertainty of impedance components. The value of covariance indicates a degree of dependence of both components. In addition it is necessary to consider, that in accordance with Eq. (1) the sampling is started with the following initial conditions

$$u(0) = U_m \sin y_u \, ,$$
$$i(0) = I_m \sin y_i \, . \tag{17}$$

Thus, covariance matrix of impedance components $\mathbf{C}_{\mathbf{z}}$ can be written as

$$\mathbf{C}_{\mathbf{z}} = \frac{2 s_z^2 w T_s}{I_m^2 \left((w(N-1)T_s)^2 - \sin^2 w(N-1)T_s \right)} \begin{bmatrix} c_{RR} & c_{RX} \\ c_{RX} & c_{XX} \end{bmatrix} \tag{18}$$

where

$c_{RR} = w(N-1)T_s + \cos(w(N-1)T_s + 2y_i)\sin w(N-1)T_s$,

$c_{RX} = -\sin(w(N-1)T_s + 2y_{i.})\sin w(N-1)T_s$,

$c_{XX} = w(N-1)T_s - \cos(w(N-1)T_s + 2y_i)\sin w(N-1)T_s$.

One can see that the elements of covariance matrix are

$$\mathbf{C}_y = \frac{s_y^2 2wT_s}{(w(N-1)T_s)^2 - \sin^2 w(N-1)T_s} \begin{bmatrix} w(N-1)T_s + \frac{1}{2}\sin 2w(N-1)T_s & -\sin^2 w(N-1)T_s \\ -\sin^2 w(N-1)T_s & w(N-1)T_s - \frac{1}{2}\sin 2w(N-1)T_s \end{bmatrix} \tag{15}$$

dependent on initial phase of the current. According to number N of recorded samples, variances of both components are oscillating around average with frequency $2f_g$. The periodical components of both variances show a phase opposition. Therefore it is possible to minimize the variances of impedance components choosing an appropriate position of acquisition time T_m for voltage and current. If variance of one of these components is minimized, the second one attains maximum.

4. NUMERICAL SIMULATIONS

The measurement algorithms have been verified by digital simulation test using MATLAB. Simulations of the influences of quantization error of the AD converter and jitters of sampling time on uncertainty processing results have been carried out. The parameters' values given in [1]-[2] were used to test the proposed algorithms. Thus, frequency of the sinusoidal forcing signal is equal to f_g=1kHz and maximum value of this voltage is 20V. For the simulation, according to [1], the 16-bit AD converter has been used. What is more, to simplify considerations, it was established that independently of values of measured impedance, the ranges of AD converters were adequate for amplitudes of sampled signals. Influence of jitters of sampling time of measured signals has been taken into consideration due to adding random (uniformly distributed) noises. The amplitude of those noises was fixed and set to $\pm 0{,}001\% T_s$. The sampling frequency is f_s=48kHz, as given in [1]-[2]. The investigations have been carried out with data window shorter then one cycle of the processed signals. Simulations were based on 1000 independent realisations. The effect of the number of samples N on the estimates of impedance components has been investigated for inductive and capacitive character of impedance. Fig.1 shows the empirical standard deviation of impedance components s_R, s_X as a function of number of samples N for indirect (solid line) and direct measurement methods (dotted line). The number of samples was being changed from N=4 to N=48. Fig.1 gives the results obtained for impedance Z_x=(1000+j1000)Ω.

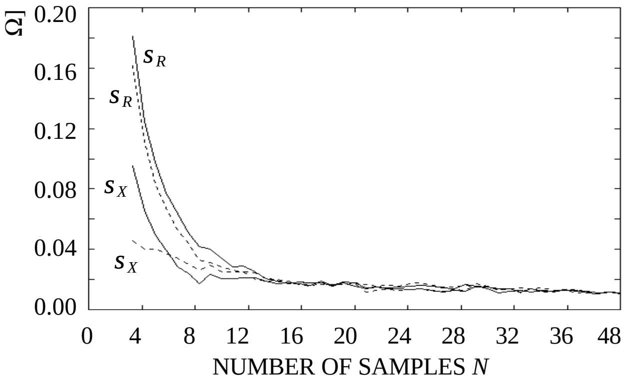

Fig.1. Standard deviations of impedance components versus number of samples

It has been shown that the number of samples greater then N=16 has no effect on the value of uncertainty for both proposed algorithms. For shorter records of data the uncertainty was increasing.
In order to compare the performance of algorithms with the results of analysis, which followed from Eq. (16) and Eq. (18), the effect of initial phase value on the standard deviations has been investigated. For both methods two different impedance, which are defined by the impedance phase angle, have been tested. The first one is inductive impedance with equal value of both components, the second one is a capacitve impedance with strongly prevailing reactance.

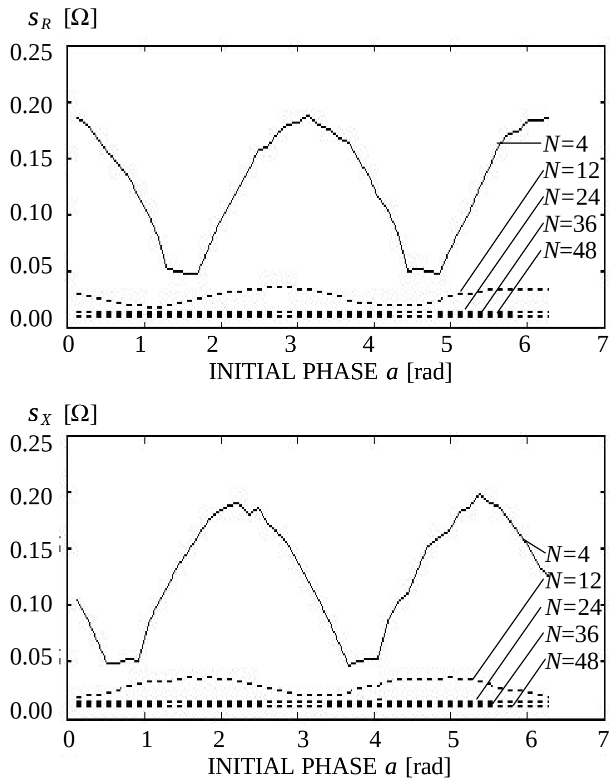

Fig.2. Standard deviations of impedance components versus initial phase of data records for indirect method (Z_x=1000+j1000Ω)

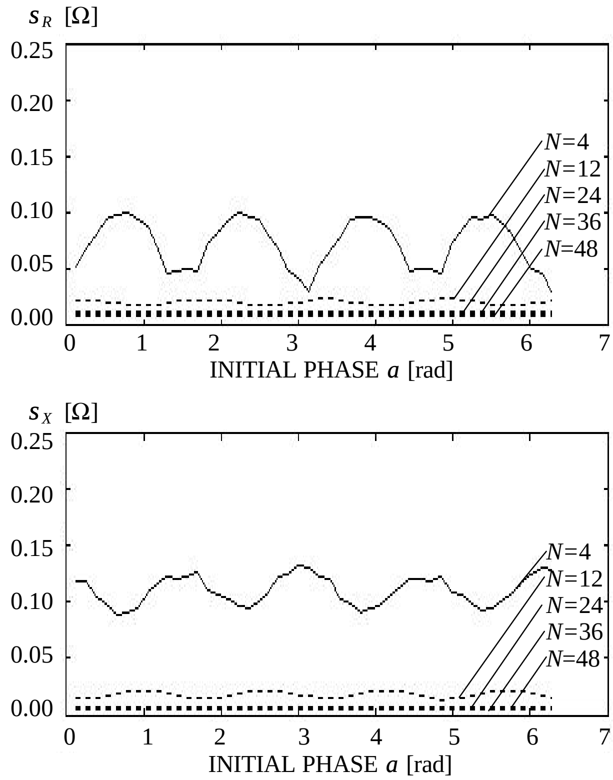

Fig.3. Standard deviations of impedance components versus initial phase of data records for indirect method (Z_x=1-j1000Ω)

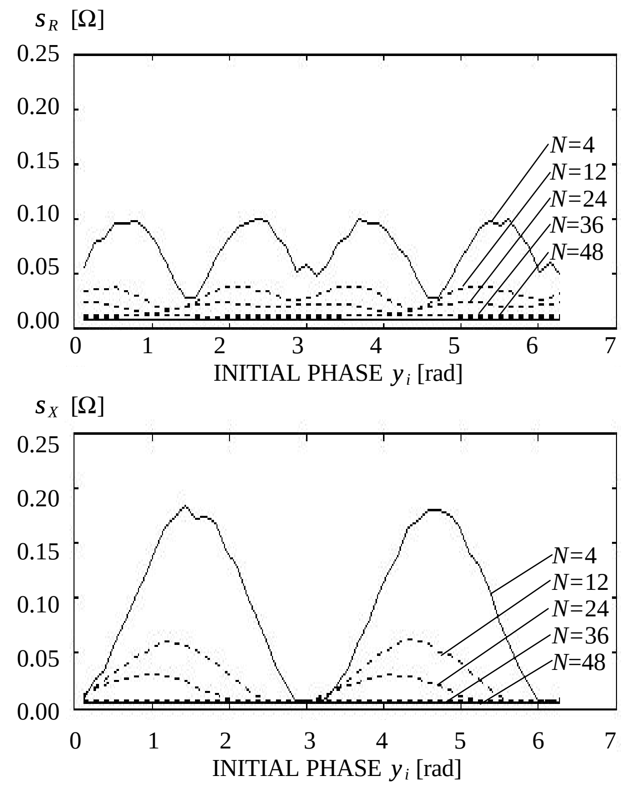

Fig.4. Standard deviations of impedance components versus initial phase of the current for direct method (Z_x=1000+j1000Ω)

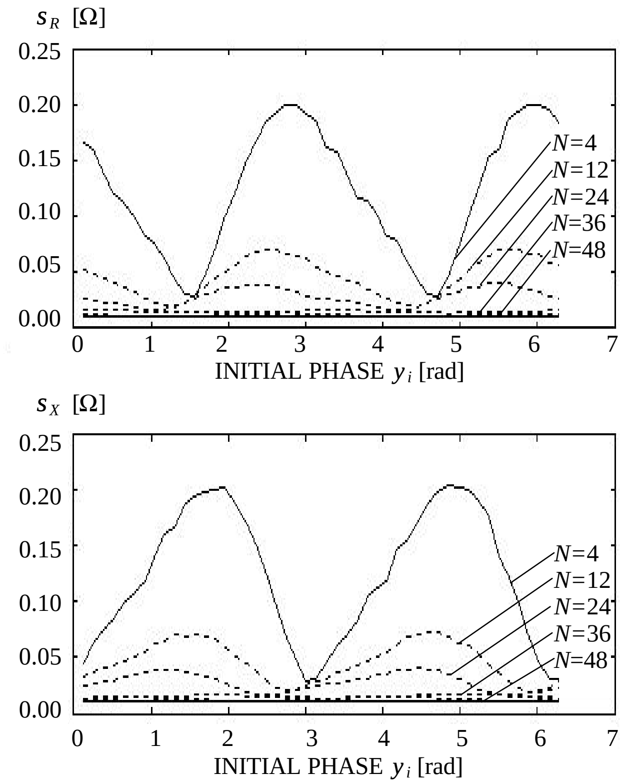

Fig.5. Standard deviations of impedance components versus initial phase of the current for direct method (Z_x=1-j1000Ω)

Fig.2 and Fig.3 show the empirical standard deviation of impedance components s_R, s_X as a function of initial phase angle a for records of N=4, 12, 24, 36, 48 samples. Similar results for direct method are displayed in Fig.4 and Fig.5, where argument of function is the initial phase of current y_i. Experimental results show that empirical standard deviations of impedance components are periodical function of initial phase. For impedance with equal value of real and imaginary part, the frequency of oscillation is equal to $2f_g$. For impedance with one of two strongly prevailing components, oscillations of frequency $4f_g$ appeared in the graph of standard deviation.

5. CONCLUSIONS

Two algorithms based on the least mean square method have been analysed. In the first one estimates of impedance components are obtained by calculating vectors of orthogonal components of voltage and current. It reconstructs indirect measurement method. The second one, on the grounds of the sequences of voltage and current samples, directly estimates values of impedance components. Both algorithms have ensured reduction of the measurement uncertainty for number of samples comparable with number of samples per period. For selected values of initial phase (different for each of impedance components) the uncertainty has been minimized. For small number of samples the minimum of uncertainty for direct methods is considerably lower. For impedance with known phase angle it is possible to select the value of initial phase to obtain minimum of uncertainty. For impedance with strongly prevailing reactance, for selected initial phase, smaller uncertainty ensured algorithm of direct method.

6. REFERENCES

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