Contribution of Interharmonic Component on the Interpolated DFT Frequency Estimator

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Abstract **– This paper investigates the contribution to the sine-wave frequency estimator returned by the interpolated discrete Fourier transform (IpDFT) algorithm of a small amplitude interharmonic component located at least one bin apart the unknown frequency. An analytical expression for the frequency estimation error is derived assuming that a maximum sidelobe decay (MSD) window is employed and its accuracy is verified through computer simulations. Based on the derived expression some useful remarks are finally drawn.**
Keywords **– Error analysis, Frequency estimation, Interharmonic component, Interpolated DFT algorithm, Windowing.**

I. INTRODUCTION

In practice, sine-waves are often non-coherently sampled and so a non-integer number of cycles is observed. Due to frequency discretization or picket-fence effect [1], only the integer part of that number can be identified by the largest discrete spectrum sample of the sine-wave, assuming that the frequency signal-to-noise ratio is at least about 18 dB [2], [3]. Conversely, the fractional part of the number of observed cycles coincides with the sine-wave inter-bin frequency location. It is often estimated by interpolating the two largest DFT spectral samples applying the so-called interpolated discrete Fourier transform (IpDFT) algorithm [1], [2], [4]-[7]. Due to non-coherent sampling, the detrimental phenomenon of spectral leakage also occurs [8], which is often reduced by weighting the acquired signal with a suitable window function. Very simple expressions for the IpDFT inter-bin frequency estimator are achieved when maximum sidelobe decay (MSD) windows are used [8], [9]. They are cosine class windows that exhibit the maximum spectrum sidelobe decay rate for a given number of cosine terms, which is strictly related with the windows spectrum mainlobe width. Due to that feature, the MSD windows are very effective in reducing the effect on the estimated frequency of long-range spectral interference due to harmonics, interharmonics or other spurious tones. However, windowing effectiveness on nearby interfering tones is greatly reduced so that their influence on frequency estimation accuracy can be relevant. An analytical expression for the inter-bin frequency estimation error that affects the IpDFT estimator based on a generic MSD window has been derived in [10] in the case of harmonically distorted sine-waves. That expression takes into account the effect of the spectral interference from both the fundamental image component and harmonics. This paper extends that result analysing the contribution of a small-amplitude interharmonic component assuming that the difference in the number of observed cycles with respect to the fundamental is greater than one bin. It is worth noticing that the assumed constraints can often be satisfied in practice. Moreover, the accuracy of the derived expression is verified through computer simulations.

II. EXPRESSION FOR THE FREQUENCY ESTIMATION ERROR FOR A SINE-WAVE AFFETCTED BY AN INTERHARMONIC

The analyzed discrete-time sine-wave affected by an interharmonic component is modeled by:

$$x(m) = \mathrm{A}\cos\left(2\pi f\left(m+\tfrac{1}{2}\right)+\phi\right) + A_{ih}\cos\left(2\pi f_{ih}\left(m+\tfrac{1}{2}\right)+\phi_{ih}\right) \quad (1)$$

$$m = -\frac{M}{2}, -\frac{M}{2}+1, \ldots, \frac{M}{2}-1$$

where A, f, ϕ, A_{ih}, f_{ih}, and ϕ_{ih} are the amplitude, normalized frequency, and initial phase of the fundamental component and the interharmonic, respectively, while M is the acquisition length, which is assumed to be an even integer number.

The fundamental and interharmonic normalized frequencies f and f_{ih} can be expressed as:

$$f \stackrel{\text{def}}{=} \frac{f_0}{f_s} = \frac{\nu}{M} = \frac{l+\delta}{M}, \quad (2)$$

$$f_{ih} \stackrel{\text{def}}{=} \frac{f_{0ih}}{f_s} = \frac{\nu_{ih}}{M} = \frac{l_{ih}+\delta_{ih}}{M}, \quad (3)$$

where f_0 and f_{0ih} are the corresponding frequencies (in Hz) and ν and ν_{ih} are the number of acquired cycles of the original continuous-time sine-waves and f_s is the sampling rate; l and l_{ih} are the integer parts of ν and ν_{ih}, respectively, and δ ($-0.5 \le \delta < 0.5$) and δ_{ih} ($-0.5 \le \delta_{ih} < 0.5$) are the corresponding fractional parts, i.e., the inter-bin frequency locations.

In practice, both the fundamental and the interharmonic components are non-coherently sampled (i.e., $\delta \neq 0$ and $\delta_{ih} \neq 0$), so that the discrete spectrum of the signal (1) exhibits spectral leakage. To reduce that phenomenon a window function $w(\cdot)$ is used and the windowed signal $x_w(m) = x(m)\cdot w(m)$ is analyzed. An H-term MSD window ($H \ge 2$) is considered in the following due to its good spectral characteristics. It is defined as:

$$w(m) \stackrel{\text{def}}{=} \sum_{h=0}^{H-1} a_h \cos\left(2\pi\frac{h}{M}\left(m+\frac{1}{2}\right)\right),$$

$$m = -\frac{M}{2}, -\frac{M}{2}+1, \ldots, \frac{M}{2}-1 \quad (4)$$

where the window coefficients a_h are given by [6]: $a_0 = \frac{C_{2H-2}^{H-1}}{2^{2H-2}}$, $a_h = \frac{C_{2H-2}^{H-h-1}}{2^{2H-3}}$, $h = 1, 2, \ldots, H-1$, in which $C_a^b = \frac{a!}{(a-b)!\ b!}$.

The DFT of the windowed signal $x_w(\cdot)$ is given by:

$$X_w(k) \stackrel{\text{def}}{=} \sum_{m=-\frac{M}{2}}^{\frac{M}{2}-1} x_w(m) e^{-j\frac{2\pi}{M}k\left(m+\frac{1}{2}\right)} = \frac{A}{2}W(k-\nu)e^{j\phi} + \frac{A}{2}W(k+\nu)e^{-j\phi} + \frac{A_{ih}}{2}W(k-\nu_{ih})e^{j\phi_{ih}} + \frac{A_{ih}}{2}W(k+\nu_{ih})e^{-j\phi_{ih}}, \quad (5)$$

where $W(\cdot)$ is the discrete time Fourier transform (DTFT) of the window $w(\cdot)$. For $|\lambda| << M$ and $M >> 1$ it is given by [6]:

$$W(\lambda) = \frac{M\sin(\pi\lambda)}{2^{2H-2}\pi\lambda}\frac{(2H-2)\ !}{\prod_{h=1}^{H-1}(h^2-\lambda^2)}. \quad (6)$$

From (6) it follows that $W(\cdot)$ is an even-symmetry function.

It should be noticed that the second and the fourth terms in (5) represent the spectral image of the fundamental and interharmonic components, respectively.

The inter-bin frequency location estimator returned by the IpDFT algorithm based on the H-term MSD window is given by [6]:

$$\hat{\delta} = \frac{(H-1+i)\alpha - H + i}{\alpha+1}, \quad (7)$$

where $i = 0$ if $|X_w(l-1)| \ge |X_w(l+1)|$ and $i = 1$ if $|X_w(l-1)| < |X_w(l+1)|$ and $\alpha \stackrel{\text{def}}{=} \frac{|X_w(l+i)|}{|X_w(l-1+i)|}$.

When $l_{ih} \ge l+1$ and $A_{ih} << A$, the inter-bin frequency location estimation error due to an interharmonic component is given by:

$$\Delta\delta \cong (H+(-1)^i\delta)\left[\frac{(-1)^i 2\nu}{2\nu-\delta+(-1)^{i+1}H}\frac{W(2\nu-\delta)}{W(-\delta)}\cos(2\phi) + \frac{(-1)^i(\nu_{ih}-\nu)}{\nu_{ih}-\nu+\delta+(-1)^i H}\frac{A_{ih}}{A}\frac{W(\nu-\nu_{ih}-\delta)}{W(-\delta)}\cos(\phi_{ih}-\phi)\right]. \quad (8)$$

The derivation of the expression (8) is given in Appendix.

In the particular case $\nu_{ih} = k\nu$, (8) returns the frequency estimation error due to the kth harmonic [10]. As we can observe, (8) have two terms exhibiting a sine-like behavior. The first one is due to the spectral image of the fundamental component and it decreases as the number of observed fundamental cycles ν increases. The second term is due to the interharmonic component and it decreases as the difference $\nu_{ih} - \nu$ between the observed cycles of interharmonic and fundamental components increases. When that difference is small the contribution of the interharmonic component can prevail even though that spectral line has a small amplitude. Obviously, the value of $\nu_{ih} - \nu$ can be increased by increasing the length of the observation interval, that is increasing the acquisition length M when a constant sampling rate is assumed.

III. COMPUTER SIMULATIONS

The accuracy of the derived expression (8) for the frequency estimation error has been verified through computer simulations. The amplitude of the simulated sine-wave wave and the acquisition length were $A = 1$ p.u. and $M = 512$, respectively. Each simulation exploits the results of 1000 runs in which the initial phases of both the fundamental component and the interharmonic were changed at random in the range [0, 2π) rad. The two-term MSD window or Hann window [9] was used.

Fig. 1 shows the maximum value of the magnitude of the relative frequency estimation error, ε_δ, obtained by simulation and theoretical results when the interharmonic amplitude is $A_{ih} = 0.1$ p.u.. The behavior of that error is analyzed with respect to the difference $\nu_{ih} - \nu$ between the numbers of interharmonic and fundamental cycles when $\nu = 1.7$, 5.7, 10.7, and 15.7 cycles (Fig. 1(a)), and $\nu = 2.3$, 6.3, 11.3, and 16.3 cycles (Fig. 1(b)). The two cases correspond to $\delta = -0.3$ and $\delta = +0.3$, respectively.

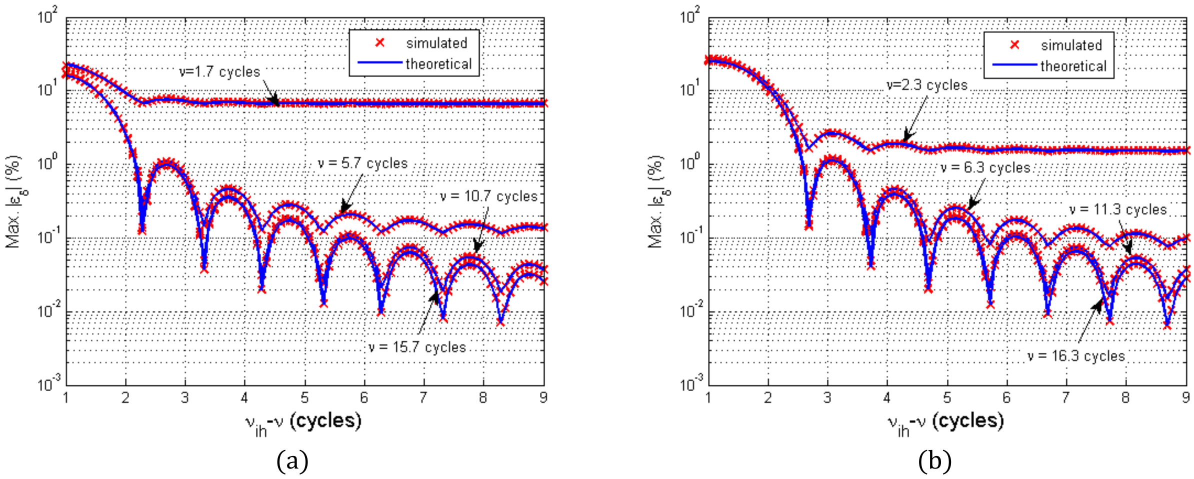


Fig. 1. Simulation and theoretical results related to the maximum magnitude of the relative error ε_δ versus the distance $\nu_{ih} - \nu$ between the interharmonic and the fundamental components when $\nu = 1.7$, 5.7, 10.7, and 15.7 cycles (a) or $\nu = 2.3$, 6.3, 11.3, and 16.3 cycles (b). $A_{ih}/A = 0.1$ p.u., 1000 runs of $M = 512$ samples each, and the Hann window are considered.

As it can be seen, the agreement between the simulation and the theoretical results is very good. Moreover, the relative frequency estimation errors are quite significant when the two tones are close to each other (i.e., when $\nu_{ih} - \nu$ is less than about 1.5 cycles) and the estimator accuracy almost depends only on the interference due to the interharmonic. Also, when $\nu_{ih} - \nu$ is sufficiently smaller than the number of observed cycles ν of the fundamental component, the effect of the spectral image becomes negligible. Conversely, the contribution of the spectral image component becomes significant when $\nu_{ih} - \nu$ increases. For instance, this occurs when $\nu_{ih} - \nu$ is higher than 3 or 3.5 cycles for $\nu = 1.7$ or 2.3 cycles, respectively.

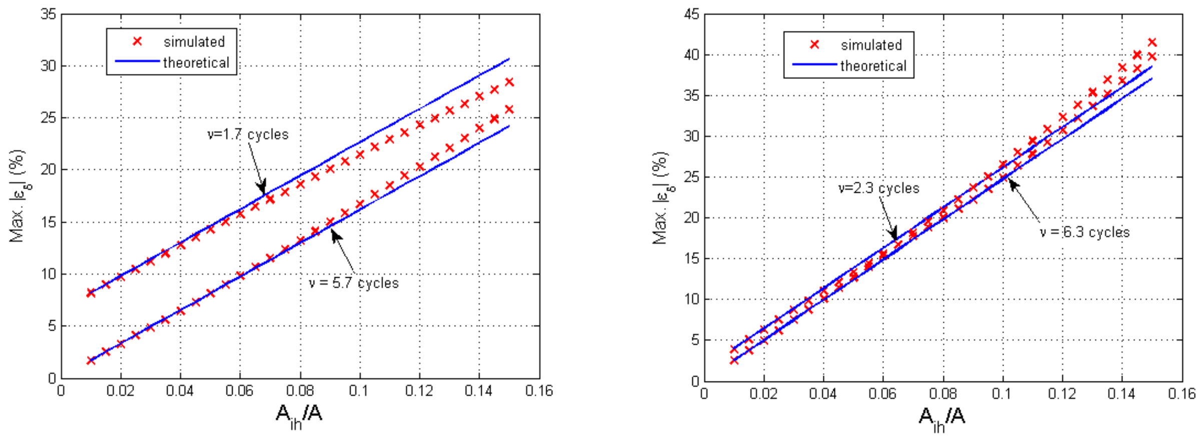
It is worth noticing that the above conclusions hold regardless the value of the number of cycles ν as soon as it is greater than 1.5 cycles, except in quasi-coherent condition (i.e., for δ very close to zero). Indeed, in such situation the simulation and theoretical results can differ significantly due to the possible wrong choice of the second interpolation point between the two samples $|X_w(l - 1)|$ and $|X_w(l + 1)|$, which exhibit very close values.

Fig. 2 shows the simulation and theoretical results related to the maximum magnitude of the relative estimation error, ε_δ, as a function of the ratio A_{ih}/A between the interharmonic and fundamental amplitudes when $\nu = 1.7$ and 5.7 cycles (Fig. 2(a)), $\nu = 2.3$ and 6.3 cycles (Fig. 2(b)) and $\nu_{ih} - \nu = 1$ cycle. The above difference is the smallest one considered.

Fig. 2 shows that the simulation and the theoretical results are very close to each other when $A_{ih}/A \leq 0.1$. It is worth noticing that the results obtained for $\nu > 5.7$ or $\nu > 6.3$ cycles are very close to those reported in Fig. 2(a) for $\nu = 5.7$ cycles and in Fig. 2(b) for $\nu > 6.3$ cycles, respectively.

Finally, Fig. 3 shows the comparison between the maximum magnitude of the frequency estimation error $\Delta\delta$ due to an interhamonic when $\nu_{ih} - \nu = 1.7$ cycles and a second harmonic as a function of ν when both interharmonic and harmonic have the same amplitude equal to 0.1 p.u. The theoretical expression for the error contribution due to the second harmonic is obtained from (8) assuming $\nu_{ih} = 2\nu$.

As it can be seen, the frequency estimation error due to the interharmonic is almost independent of the integer part of the number of observed sine-wave cycles and it depends mainly on the inter-bin frequency location δ. Indeed, that contribution is related to the second term in (8), which, for fixed values of $\nu_{ih} - \nu$, depends only on δ. Therefore, the magnitude of the error ε_δ has an almost periodic behavior with respect to ν. Conversely, the frequency estimation error due to the effect of the second harmonic decreases as the number of observed sine-wave cycles ν increases. Moreover, it is dominated by the effect of interharmonic, except when $\nu < 2$ cycles, where the contribution of the spectral image component becomes significant.



(a) (b)

Fig. 2. Simulation and theoretical results related to the maximum magnitude of the relative frequency estimation error ε_δ versus the ratio A_{ih}/A when $\nu = 1.7$ and 5.7 cycles (a), or $\nu = 2.3$ and 6.3 cycles (b). $\nu_{ih} - \nu = 1$ cycle, 1000 runs of $M = 512$ samples each, and the Hann window are considered.

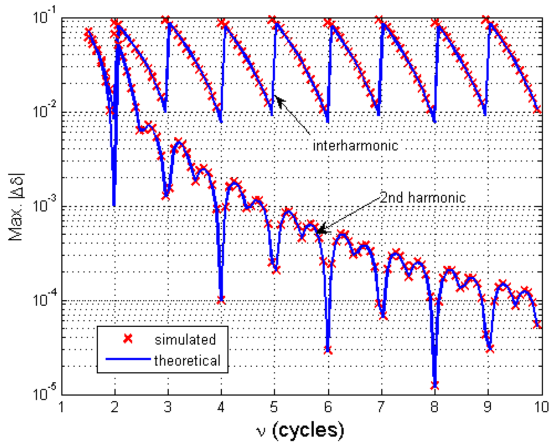


Fig. 3. Simulation and theoretical results related to the maximum magnitude of the frequency estimation error $\Delta\delta$ versus ν when the sine-wave is affected by both interharmonic and second harmonic of the same amplitude equal to 0.1 p.u.. The interharmonic is 1.7 cycles from the fundamental component. $A = 1$ p.u., 1000 runs of $M = 512$ samples each, and the Hann window are considered.

IV. CONCLUSIONS

In this paper an analytical expression for the inter-bin frequency location estimation error returned by the IpDFT algorithm based on a generic MSD window has been derived in the case of sine-wave affected by a small-amplitude interharmonic located more than one bin apart from the unknown frequency. The accuracy of the derived expression has been confirmed through computer simulations. It has been shown that the frequency estimation error can be significant when the frequency separation between the interharmonic and the sine-wave is small and it can be much greater than the effect due to the spectral interference from a second harmonic of equal amplitude.

APPENDIX

Derivation of the expression for the error $\Delta\delta$

From (5), after simple calculations, we obtain:

$$|X_w(l+r)|^2 = \frac{A^2}{4}\left[W^2(r-\delta) + W^2(2\nu-\delta+r) + 2W(r-\delta)W(2\nu-\delta+r)\cos(2\phi)\right] + \frac{A_{ih}^2}{4}\left[W^2(\nu-\nu_{ih}-\delta+r) + W^2(\nu+\nu_{ih}-\delta+r) + 2W(\nu-\nu_{ih}-\delta+r)W(\nu+\nu_{ih}-\delta-r)\cos(2\phi_{ih})\right] + \frac{A\cdot A_{ih}}{2}\left[W(r-\delta)W(\nu-\nu_{ih}-\delta+r)cos(\phi-\phi_{ih}) + W(r-\delta)W(\nu+\nu_{ih}-\delta+r)cos(\phi+\phi_{ih}) + W(2\nu-\delta+r)W(\nu-\nu_{ih}-\delta+r)cos(\phi+\phi_{ih}) + W(2\nu-\delta+r)W(\nu+\nu_{ih}-\delta+r)cos(\phi-\phi_{ih})\right], r = -1, 0, 1 \quad \text{(A.1)}$$

When $l_{ih} \geq l + 1$ and $A_{ih} << A$, $|X_w(l+r)|^2$ can be approximated as

$|X_w(l+r)|^2 \cong \frac{A^2}{4}[W^2(r-\delta)+2W(r-\delta)W(2\nu + -\delta+r)\cos(2\phi)] + \frac{A \cdot A_{ih}}{2} W(r-\delta)W(\nu-\nu_{ih}-\delta + r)\cos(\phi - \phi_{ih}),$

(A.2)

since the above terms are much higher than the other terms in (A.1).

From (A.2), based on the approximation $\sqrt{1+x} \cong 1+\frac{x}{2}$, when $|x| << 1$, from (A.2) we achieve:

$$|X_w(l+r)| \cong \frac{A}{2} W(r-\delta)\left[1 + \frac{W(2\nu-\delta+r)}{W(r-\delta)}\cos(2\phi) + \frac{A_{ih}}{A}\frac{W(\nu-\nu_{ih}-\delta+r)}{W(r-\delta)} cos(\phi - \phi_{ih})\right], \quad (A.3)$$

From (6) the following equalities can be achieved:

$$W((-1)^{i+1}-\delta) = \frac{H-1+(-1)^{i+1}\delta}{H+(-1)^i\delta} W(-\delta),$$

$$W(2\nu-\delta+(-1)^{i+1}) = -\frac{2\nu-\delta+(-1)^i(H-1)}{2\nu-\delta+(-1)^{i+1}H} W(2\nu-\delta),$$

$$W(\nu-\nu_{ih}-\delta+(-1)^{i+1}) = \frac{H-1+(-1)^i(\nu-\nu_{ih}-\delta)}{H+(-1)^{i+1}(\nu-\nu_{ih}-\delta)} W(\nu-\nu_{ih}-\delta). \quad (A.4)$$

Based on (A.3) and (A.4), after some calculations we obtain:

$$\alpha = \frac{|X_w(l+i)|}{|X_w(l-1+i)|} \cong \frac{W(-\delta+i)}{W(-1-\delta+i)}\left(1 + \frac{2H-1}{H+(-1)^{i+1}\delta-1}\gamma\right), \quad (A.5)$$

where

$$\gamma = \frac{2(-1)^i\nu}{2\nu-\delta+(-1)^{i+1}H}\frac{W(2\nu-\delta)}{W(-\delta)} cos(2\phi) + \frac{A_{ih}}{A}\frac{(-1)^i(\nu_{ih}-\nu)}{\nu_{ih}-\nu+\delta+(-1)^iH}\frac{W(\nu-\nu_{ih}-\delta)}{W(-\delta)} cos(\phi - \phi_{ih}). \quad (A.6)$$

From (7) it follows that the error $\Delta\delta$ is given by [10]:

$$\Delta\delta = (H+(-1)^i\delta)\gamma. \quad (A.7)$$

By replacing (A.6) into (A.7) the expression (8) is finally achieved.

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