

# Possibility distribution: a new mathematical tool to express measurement uncertainty

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**Abstract** – In the last years, the Authors have proposed the Theory of Evidence as the mathematical framework to handle measurement uncertainty, and the possibility distribution as the mathematical variable to represent it. The mathematical Theory of Evidence is more general than the well known probability theory and appears to be more suitable to represent all possible kinds of uncertainty contributions. The aim of this paper is to give a general overview of this theory, showing its potentialities and the obtained results.

## I. INTRODUCTION

Since the Guide to the Expression of Uncertainty in Measurement (GUM) [1] has been issued, measurement uncertainty has been defined, evaluated and expressed following a strict probabilistic approach.

This probabilistic framework shows some limitations, and, during the recent years, an approach based on the more general theories of evidence and possibility [2] has been proposed by several authors to overcome these limitation [3–5]. This paper is aimed at providing the very fundamentals of this new approach, to show that it can be an useful generalization of the one proposed by the GUM.

## II. BASIC PRINCIPLES OF THE THEORY OF EVIDENCE

The Theory of Evidence has been defined by Shafer, and presented for the first time in his book in 1976 [2]. This mathematical theory is based on the definition of two functions: the belief function and the plausibility function. The meaning assigned by Shafer to these functions is here reported [2]. As far as the belief function is concerned, he wrote: *whenever I write of the "degree of support" that given evidence provides for a proposition or of the "degree of belief" that an individual accords to the proposition, I picture in my mind an act of judgment. I do not pretend that there exists an objective relation between given evidence and a given proposition that determines a precise numerical degree of support. Nor do I pretend that an actual human being's state of mind with respect to a proposition can ever be described by a precise real number called his degree of belief. Rather, I merely suppose that an individual can make a judgment ... he can announce a number that represents the degree to which he judges that evidence*

*to support a given proposition and, hence, the degree of belief he wishes to accord the proposition.* As far as the plausibility function is concerned, he wrote: *it expresses the extent to which one fails to doubt A - i.e., the extent to which one finds A credible or plausible, being A the considered proposition.*

Although Shafer's belief and plausibility functions are acts of personal judgments, and hence their values depend on the actual human being, they obey to strict mathematical rules. Let  $X$  be the universal set and  $P(X)$  its power set<sup>1</sup>. The belief function is defined as [2, 6, 7]:

$$Bel(X) : P(X) \rightarrow [0, 1] \quad (1)$$

such that:

$$Bel(\emptyset) = 0; Bel(X) = 1 \quad (2)$$

and, for any  $A, B \in P(X)$ :

$$Bel(A \cup B) \geq Bel(A) + Bel(B) - Bel(A \cap B)^2 \quad (3)$$

while the plausibility function is defined by [2, 6, 7]:

$$Pl(X) : P(X) \rightarrow [0, 1] \quad (4)$$

such that:

$$Pl(\emptyset) = 0; Pl(X) = 1 \quad (5)$$

and, for any  $A, B \in P(X)$ :

$$Pl(A \cap B) \leq Pl(A) + Pl(B) - Pl(A \cup B) \quad (6)$$

Eq. (3) is called **superadditivity rule**. It is interesting to consider the case of complementary sets  $A$  and  $A^c$ , so that eq. (3), (6) become, for every  $A \in P(X)$ :

$$Bel(A) + Bel(A^c) \leq 1; Pl(A) + Pl(A^c) \geq 1 \quad (7)$$

It is also interesting to compare eq. (7) with the similar formula for probability functions, called **additivity rule**:

$$Pro(A) + Pro(A^c) = 1 \quad (8)$$

Eqs. (7), (8) suggest that probability functions obey to a stricter rule: once the probability of an event is given, the

<sup>1</sup>The power set of any set  $X$  is the set of all subsets of  $X$ , including  $X$  itself and the empty set  $\emptyset$ .

<sup>2</sup>Only two sets are here considered for the sake of brevity. The general formula for  $n$  sets can be found in [2, 6, 8].

probability of the complementary event is set; on the other hand, being belief and plausibility *acts of personal judgments*, two independent degrees of belief (and plausibility) can be associated to  $A$  and  $A^c$ . Eqs. (7), (8) also yield [2]:

$$Bel(A) \leq Pro(A) \leq Pl(A) \quad (9)$$

and this is the reason why belief and plausibility functions are also called, respectively, **lower** and **upper probability functions**.

Eqs. (7)-(9) show how the belief and plausibility functions generalize the probability functions and represent one of the reasons for stating that the Theory of Evidence is a generalization of the probability theory. A second, more important reason supporting this statement is related to the definition of another function, called *basic probability assignment function* [2, 6–8]:

$$m : P(X) \rightarrow [0, 1]; m(\emptyset) = 0; \sum_{A \in P(X)} m(A) = 1 \quad (10)$$

Since  $m(A)$  represents the degree of belief that an element  $x$  belongs to set  $A$ , it follows [2, 6–8]:

$$Bel(A) = \sum_{B|B \subseteq A} m(B); Pl(A) = \sum_{B|A \cap B \neq \emptyset} m(B) \quad (11)$$

An important concept in the Theory of Evidence is given by the focal elements, that is, those sets in  $P(X)$  where the degree of belief focuses. In mathematical terms, a set  $A \in P(X)$  is a focal element if  $m(A) > 0$  [2, 6–8].

#### A. Probability theory within the Theory of Evidence

Let us consider the case where all focal elements are singletons, that is:  $m(\{x\}) > 0$  for every  $x \in X$ . It can be proved [2, 6] that, under this assumption:

- the belief functions obey to eq. (8);
- eq. (9) becomes  $Bel(A) = Pro(A) = Pl(A)$ ;
- the basic probability assignment function is a probability distribution function  $p$ , that is:  $m(\{x\}) = p(x)$ .

This proves that the probability theory is a particular case of the Theory of Evidence. Therefore, if an approach to measurement uncertainty based on the Theory of Evidence is followed, it represents a generalization of the actual approach [6–8].

#### B. Possibility theory

A second particular case of the Theory of Evidence is called possibility theory, and is obtained when the focal elements are nested sets. Let us consider the universal set  $X = \{x_1, x_2, \dots, x_n\}$ , and the focal elements  $A_i =$

$\{x_1, x_2, \dots, x_i\}$ , that is  $m(A_i) > 0$ . Then, the focal elements are all nested:  $A_1 \subset A_2 \subset \dots \subset A_i \subset \dots \subset A_n = X$ <sup>3</sup>. The nested form of the focal elements yields a simplification of eqs. (11); moreover, the belief and plausibility functions are called necessity (*Nec*) and possibility (*Pos*) functions respectively [2, 6–8]:

$$Nec(A_j) = \sum_{k=1}^j m(A_k); Pos(A_j) = 1 \quad (12)$$

Possibility functions *Pos* are fully represented by a distribution function, called possibility distribution function  $r$  and defined as [2, 6–8]:

$$r(x) = Pos(\{x\}); Pos(A) = \sup_{x \in A} r(x) \quad (13)$$

where

$$r : X \rightarrow [0, 1]; \sup_{x \in X} r(x) = 1 \quad (14)$$

The relationship between the basic probability assignment function  $m$  and the possibility distribution function  $r$  is [2, 6–8]:

$$r_i = \sum_{k=i}^n m_k; m_i = r_i - r_{i+1} \quad (15)$$

where  $r_{n+1} = 0$  by definition and where  $r_i = r(x_i)$  and  $m_i = m(A_i)$ .

#### C. Shafer's total ignorance

Let us now recall Shafer's concept of **total ignorance**, that is: *the situation where we have no evidence about  $X$  at all* [2, 6, 8]. In other words, we have no further knowledge about the considered event, except that it belongs to  $X$ .

Total ignorance can be mathematically represented by the following belief, plausibility and basic probability assignment functions [8]:

$$Bel(X) = 1; Bel(A) = 0 \quad \forall A \in P(X) \quad (16)$$

which shows that full belief is associated to the universal set  $X$  but no belief is associated to any of its subsets;

$$Pl(A) = 1 \quad \forall A \in P(X) \quad (17)$$

which shows that any subset of  $X$  is totally plausible;

$$m(X) = 1; m(A) = 0 \quad \forall A \neq X \quad (18)$$

It is now interesting to ask oneself whether total ignorance can be represented also within the probability and possibility theories.

<sup>3</sup>From the theoretical point of view, an infinite universal set and infinite number of focal elements should be considered for the sake of generality. However, as it will be shown later in this paper, the focal elements are the cuts of given distributions. In practice, a discrete, finite number of these cuts is processed. Hence, the assumption of a finite number of focal elements does not affect generality.

If total ignorance could be represented within the probability theory, then there should be a probability distribution function  $p$  representing, mathematically, the concept of total ignorance. However, if we consider a probability distribution, we are no more in the situation where *we have no evidence about X at all*, because we add the knowledge of this probability distribution and we add the constraint that the focal elements are singletons. This constraint conflicts with (18), which does not allow one to assign a basic probability  $m$  greater than zero to any subset of  $X$ . Therefore, Shafer's total ignorance cannot be represented within the probability theory. No probability distribution can mathematically represent Shafer's total ignorance correctly and, if a probability distribution is assigned over  $X$ , this situation cannot be considered as total ignorance any longer.

If total ignorance could be represented within the possibility theory, then there should be a possibility distribution function  $r$  representing, mathematically, the concept of total ignorance. Again, at a first extent, if we consider a possibility distribution, we may think that we are not in *the situation where we have no evidence about X at all*, because we are adding the constraint that the focal elements are all nested. On the other hand, this different assumption does not conflict with (18). In fact, the assumption that the focal elements are nested ( $A_1 \subset A_2 \subset \dots \subset A_i \subset \dots A_n = X$ ) does not impose the number of the focal elements: there could be  $n$ , but also fewer and, in the extreme case, only one, as in (18), where the only focal element is  $X$ . Therefore, it is possible to represent total ignorance within the possibility theory. This implies that a possibility distribution  $r$  exists over  $X$ , representing total ignorance. Eq. (15) yields the rectangular possibility distribution:

$$r_i = r(x_i) = 1 \quad \forall i \quad (19)$$

This is the greatest possibility distribution: all elements of the universal set  $X$  are equally possible, with no preference for one element with respect to the others.

Hence, we arrive to Shafer's firm conviction that probability functions are not able to handle correctly the case of total ignorance, while possibility functions are. The common use of an uniform probability distribution to represent total ignorance is not representative of Shafer's total ignorance, *the situation where we have no evidence about X at all*, since it represents the situation where the evidence shows that every element of  $X$  has the same probability.

#### D. Fuzzy variables

Fuzzy Variables (FVs) have been recently employed in several applications of fuzzy logic, mainly in the field of automatic controls. However, it is not generally known that the fuzzy variable is the natural variable in the possibility theory, exactly in the same way as the random variable is the natural variable in the probability theory, as here briefly shown.

An FV  $A$  is defined by its membership function [9]:

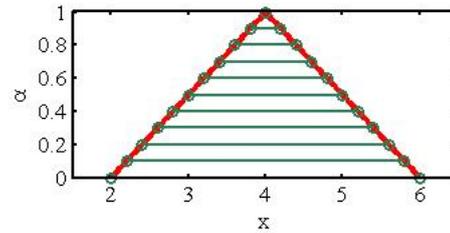


Fig. 1. Example of fuzzy variable and some of its  $\alpha$ -cuts at levels  $\alpha = 0, \alpha = 0.1, \dots, \alpha = 1$ .

a convex function  $\mu_A(x) : X \rightarrow [0, 1]$  such that  $\sup_X \mu_A(x) = 1$ . It can be proved [10] that this membership function is a possibility distribution ( $\mu_A(x) = r_A(x)$ ), thus showing that the FVs are mathematically defined within the possibility theory. An alternative way to define an FV is to provide its  $\alpha$ -cuts  $A_\alpha$  for every level  $\alpha \in [0, 1]$  [6, 9]:

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\} \quad (20)$$

Eq. (20) means that the  $\alpha$ -cuts of an FV are intervals defined in such a way that, if  $\alpha_1 > \alpha_2$ , then  $A_{\alpha_1} \subset A_{\alpha_2}$ . Therefore, the  $\alpha$ -cuts of an FV are nested intervals. Fig. 1 shows an example of FV and some of its  $\alpha$ -cuts.) The nested form of the  $\alpha$ -cuts of an FV recalls the nested form of the focal elements (since we are in the framework of the possibility theory). As a matter of fact, given an interval  $X$  representing the universal set and given an FV over  $X$ , the  $\alpha$ -cuts of the FV are the focal elements of  $X$  [6], that is, those intervals where the degree of belief focuses, that is, where  $m(A) > 0$ . Then, (12) implies that a necessity value can be associated to each single  $\alpha$ -cut of the FV. Hence, finally, it can be stated that an FV is a set of nested intervals to any of which a necessity value, or degree of belief, is associated. It can be proved [6] that the degree of belief associated to the  $\alpha$ -cut  $A_\alpha$  is:

$$Bel(A_\alpha) = Nec(A_\alpha) = 1 - \alpha \quad (21)$$

Although the membership function (possibility distribution) of an FV is a continuous function, so that an infinite number of  $\alpha$ -cuts can be defined, in practical applications it can be described by a finite number  $M$  of  $\alpha$ -cuts<sup>4</sup>. Since an interval can be fully represented by its extremes,  $A_\alpha = [a_l^\alpha, a_r^\alpha]$ , a simple way to describe and process an FV is hence to consider and save its  $\alpha$ -cuts in an  $[M \times 2]$  matrix. In this matrix, the row index provides the level  $\alpha$  and the row elements provide the extremes of the  $\alpha$ -cut [6]:

$$A = \begin{bmatrix} a_l^{\alpha=0} & a_r^{\alpha=0} \\ a_l^{\alpha=(1/M-1)} & a_r^{\alpha=(1/M-1)} \\ a_l^{\alpha=(2/M-1)} & a_r^{\alpha=(2/M-1)} \\ \dots & \dots \\ a_l^{\alpha=1} & a_r^{\alpha=1} \end{bmatrix} \quad (22)$$

<sup>4</sup> $M = 101$  represents a good compromise between resolution and execution time.

### III. MEASUREMENT RESULTS IN TERMS OF POSSIBILITY DISTRIBUTIONS

It is well known that, when we measure a physical quantity (measurand), the obtained measured value is not, generally, the true value of the measurand, because of the presence of some influence quantities, due to the environmental conditions, the measurement instrument, and so on. Therefore, the measured value must be provided together with *an interval about the measurement result that may be expected to encompass a large fraction of the distribution of values that could reasonably be attributed to the quantity subject to measurement* [1]. *The ideal method for evaluating and expressing measurement uncertainty should be capable of readily providing such an interval, in particular, one with a coverage probability or level of confidence that corresponds in a realistic way to that required* [1].

This interval is, from the probabilistic point of view, a confidence interval, and the associated level of confidence represents the probability that the true value of the measurand falls within this interval. As stated in the above section, an FV provides a set of nested intervals with associated necessity values, or degrees of belief. It can be stated that each  $\alpha$ -cut and its necessity value extends the probabilistic concept of confidence interval and level of confidence. Therefore, a measurement result can be represented in terms of an FV, thus providing all confidence intervals and associated levels of confidence [6, 7]. This also means that the measurement result is represented in terms of a possibility distribution  $r$ .

#### A. Systematic contributions to uncertainty

A systematic contribution to uncertainty is a *component of the measurement error that, in replicate measurements, remains constant or varies in a predictable manner* [11]. Systematic significant effects are required to be identified and compensated for [1]. However, in many practical situations, the identification and compensation procedure may not be cost-effective, and only an interval of possible values is identified for this effect, generally obtained from the manufacturer specifications, calibration data or *a-priori* knowledge.

If also a probability distribution (*pdf*) is assigned over this interval, then, it is possible to convert this *pdf* into an equivalent possibility distribution (PD) by using an appropriate probability-possibility transformation, where all confidence intervals and associated levels of confidence are maintained [12, 13]. However, in general, the given interval is the only available information, and no *pdf* is known. This general practical situation corresponds exactly to Shafer's total ignorance (the situation where *we have no evidence about X at all*): we have an interval ( $X$ ), but we have no further knowledge about  $X$ . Hence, as recalled in Sec. IIC., this situation is mathematically represented by a rectangular PD.

#### B. Random contributions to uncertainty

A random contribution to uncertainty is a *component of the measurement error that, in replicate measurements, varies in an unpredictable manner* [11]. Generally, when a random contribution affects the measurement procedure, a *pdf* can be associated to the measurement result, [1, 14]. Once again, this *pdf* can be converted into its equivalent PD [12, 13]. The reason for converting the *pdf* into a PD is that, in this way, both random and systematic contributions to uncertainty can be represented with the same mathematical variable (the PD) and it is more convenient, from the practical point of view, to handle two PDs than one PD and one *pdf*.

On the other hand, from the strict mathematical point of view [2, 6, 7, 15], it is not possible to handle both random and systematic contributions to uncertainty with two *pdfs*, since, as recalled in Sec. IIC., total ignorance cannot be represented by any *pdf*. If total ignorance should be considered within the mathematical framework of the probability theory, the correct procedure would require to consider the whole family of *pdfs* over the given interval, thus yielding to a very complex procedure [15]. Moreover, within the possibility theory, eq. (9) becomes  $Nec(A) \leq Pro(A) \leq Pos(A)$  and the possibility function is the upper probability function representing the whole family of possible *pdfs* over  $X$ . Hence, it can be proved that the use of the possibility function (and hence the passage from the probability theory to the possibility theory) represents the strict mathematical way to handle this case [15].

#### C. Random-Fuzzy variables

In general, a measurement procedure is affected by both random and systematic contributions to uncertainty. The PD representing the measurement result should be hence suitably built in order to consider the effects of both these kinds of contributions. This solution, however, is not optimal if different measurement results must be combined together. In fact, systematic and random contributions to uncertainty physically combine in different ways, and, as it will be seen in the next section, these different ways are not reflected into the combined result if a single PD is used to represent, in each measurement result, both systematic and random effects. The solution to this problem requires to associate two PDs to each measurement result: a PD  $r_{ran}$ , representing the effects of the sole random contributions on the measurement result, and a PD  $r_{sys}$ , representing the effects of the sole systematic contributions on the measurement result.  $r_{ran}$  and  $r_{sys}$  are built, of course, according to the available information [6, 16], as briefly recalled in Secs. IIIB. and IIIA. These two PDs are then linearly combined to obtain a random-fuzzy variable (RFV), as shown in Fig. 2 [5, 6]. The internal PD of the RFV is  $r_{sys}$ , while the external PD is obtained from the linear combination of  $r_{sys}$  and  $r_{ran}$ .

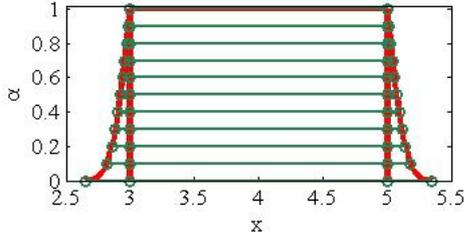


Fig. 2. Example of random-fuzzy variable and some of its  $\alpha$ -cuts at levels  $\alpha = 0, \alpha = 0.1, \dots, \alpha = 1$ .

Fig. 2 also shows some  $\alpha$ -cuts of the RFV. Each of these  $\alpha$ -cuts is a confidence interval of type 2  $A_\alpha = [a_1^\alpha, a_2^\alpha, a_3^\alpha, a_4^\alpha]$  [6, 7], thus providing the information about the effects of the systematic contributions (internal interval  $[a_2^\alpha, a_3^\alpha]$ ) and the effects of the random ones (union of the side intervals  $[a_1^\alpha, a_2^\alpha]$  and  $[a_3^\alpha, a_4^\alpha]$ ).

Extending the considerations given in Sec. IID., an RFV can be simply represented by an  $[M \times 4]$  matrix [6].

#### IV. COMBINATION OF MEASURED VALUES

Since the random and systematic contributions to uncertainty combine in different ways, also the two PDs  $r_{ran}$  and  $r_{sys}$  shall be mathematically combined in a different way, to reflect the different behavior of the contributions they represent.

A quite simple and immediate mathematics has been defined in [6, 17], that provides good approximated results. A stricter mathematical approach has been defined in [18, 19], based on the construction of the joint possibility distribution of the two initial PDs and the application of Zadeh's extension principle. The shape of the joint PD depends on: the initial PDs; the nature of the uncertainty contributions the PDs are representing; the degree of dependency of the contributions. It is not possible to enter here the mathematical details for the sake of brevity. Let us only say that, even if this strict procedure looks complex when described with equations, it can be implemented in a quite simple way, since the operations between PDs become algebraic operations between intervals (the  $\alpha$ -cuts), which are straightforward to implement and fast to compute. The readers are invited to access this web page: <http://131.175.120.11:8000/RFVcalculator.html>. The front panel of a remotely controlled application is shown, which allows the readers to create two RFVs (with various shapes and different degrees of dependence) and combine them according to the four arithmetic operations. This simple example shows that the implementation of this new approach is immediate and the result are readily obtained. When only random contributions are considered, the same results as those provided by probability are obtained, but much faster computations than Monte Carlo simulations or complicated convolutions integrals are required, as the reader can readily check.

#### V. COMPARISON OF MEASURED VALUES

The final aim of a measurement procedure is not, generally, the evaluation of the final measurement result, but the comparison of this result with a given threshold, so that a decision (with important economical, social or cultural implications) can be taken. When measurement uncertainty is taken into account, this comparison is not immediate, in particular when the given threshold is not a crisp value but a measurement result itself.

The representation of the measurement results in terms of RFVs makes this comparison feasible in a simple and immediate way [6, 20]. It is not possible to enter here the mathematical details for the sake of brevity, so we confine this point to an example. Let us consider the two RFVs  $A$  (blue lines) and  $B$  (red lines) in Fig. 3.

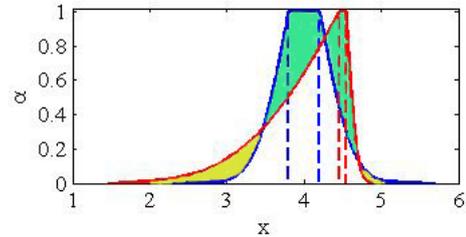


Fig. 3. Comparison of two random-fuzzy variables.

The comparison procedure of  $A$  and  $B$  allows us to define three coefficients:  $C_{A>B}$ , representing the credibility coefficient that can be associated to the statement  $A$  is greater than  $B$ ;  $C_{A<B}$ , representing the credibility coefficient that can be associated to the statement  $A$  is lower than  $B$ ;  $C_{A=B}$ , representing the credibility coefficient that can be associated to the statement  $A$  is equal to  $B$ . Of course, it is [6, 20]:

$$C_{A>B} + C_{A<B} + C_{A=B} = 1 \quad (23)$$

When the two RFVs in Fig. 3 are considered, this procedure yields:  $C_{A>B} = 0.1008$ ,  $C_{A<B} = 0.3147$ ,  $C_{A=B} = 0.5845$ , whose values are graphically represented, respectively, by the yellow, green and white areas in Fig 3.

#### VI. CONCLUSIONS

This paper has proposed a quick overview on the theories of evidence and possibility, and shown that measurement uncertainty can be expressed and processed in terms of possibility distributions, instead of probability distributions. [21–26] show some examples of applications.

It is worth reinforcing that the approach here proposed shares the same principles of the GUM, though it employs a more general mathematics. For this reason, we believe it may be usefully employed to overcome the limitations of the GUM approach and we hope this paper may trigger a fruitful discussion among the measurement experts.

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