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Estimating Filtering Errors Using the Peano Kernel Theorem

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Abstract- The Peano Kernel Theorem is introduced and a frequency domain derivation is given. It is demonstrated that the application of this theorem yields simple and accurate formulas for estimating the error introduced into a signal by filtering it to reduce noise.

I. Introduction

This paper deals with the following problem: A signal y(t) is measured with

$$y(t) = x(t) + n(t), \tag{1}$$

where $x(t)$ is the true signal and $n(t)$ is the noise. Actual measurements are made with discrete time, but continuous time will be used here, because the mathematics is more straightforward. It is assumed that the sampling density is high enough that the difference between the integrals given here and the corresponding sums is sufficiently small. The signal, x, is estimated by filtering y to reduce the effects of noise. In other words

$$\hat{x}(t) = \tilde{g} * y(t) = \tilde{g} * x(t) + \tilde{g} * n(t)\,, \tag{2}$$

where the asterisk denotes convolution, $\hat{x}(t)$ is the estimate of $x(t)$, and $\tilde{g}(t)$ is the impulse response of the filter. The reason for the tilde over the function g(t) will be explained later. Although this example uses convolution, the analysis in the paper applies equally well to time-varying filters. The error is the difference between the true signal and its estimate. It can be decomposed as follows:

$$\hat{x}(t) - x(t) = \tilde{g} * x(t) - x(t) + \tilde{g} * n(t) = (\tilde{g} - \delta) * x(t) + \tilde{g} * n(t) = e_F(t) + e_R(t)\,, \tag{3}$$

where δ is the Dirac delta function. The second term above, e_R, is the random error due to the noise, and its magnitude can be easily estimated from knowledge of the noise spectrum and the impulse response of the filter [1],[2]. This paper is concerned with estimating $e_F(t)$, the filtering error. The estimation is via the Peano Kernel Theorem (PKT) [3],[4]. Although the PKT has been used in numerical analysis for a long time, it has not been used for estimating the errors due to filtering in measurement applications. To apply the PKT, the problem must be reformulated in terms of functionals. This is done by concentrating on a particular value, $t = t_0$, and estimating $e_F(t_0)$. Without any loss of generality, we can take $t_0 = 0$, because any other value can be obtained by translation of the data. The reformulated problem is to estimate the error in approximating

$$F(x) = \int f(t)x(t)dt \text{ with } G(x) = \int g(t)x(t)dt. \tag{4}$$

In the above example f is the delta function, and g is the time reversal of the impulse response of the filter, i.e., $g(t) = \tilde{g}(-t)$. Since the function, $g(t)$, is used more often, as in (4) above, the more complex notation is applied to the impulse response. In this paper the phrase, "a filter, g," will mean a filter for which g is the time reversal of its impulse response. The examples will use zero-phase-shift filters for which the impulse response and its time reversal are the same, i.e., $g(t) = g(-t)$.

II. The Peano Kernel Theorem

A. Statement and proof of theorem

The PKT deals with approximations of the type given in (4). Writing $E(x) = G(x) - F(x)$, it asserts that

$$\begin{gathered}\text{If } E(x)=0 \text{ whenever } x \text{ is a polynomial of degree } n-1 \text{ or less,}\\ \text{then there is a function } k_n(t) \text{ such that } E(x)=\int k_n(t)x^{(n)}(t)dt.\end{gathered} \tag{5}$$

The function, $x^{(n)}(t)$, is the n^{th} derivative of $x(t)$. Formulas for calculating $k_n(t)$ are given in [2] and 0, but a different, frequency domain, approach will be used here.

The condition of the theorem, that $E(x) = 0$ for x a polynomial of degree n-1 or less, is equivalent to

$$\int g(t)dt = 1, \text{ and } \int t^k g(t) = 0 \text{ for } 1 \le k \le n-1. \tag{6}$$

Unless otherwise stated, we are assuming that $f(t) = \delta(t)$. Condition (6) can be stated in the frequency domain as

$$\hat{g}(0) = 1, \text{ and } \hat{g}^{(k)}(0) = 0 \text{ for } 1 \le k \le n-1. \tag{7}$$

The formula for $E(x)$ can be written in the frequency domain using Parseval's relation:

$$\begin{aligned} E(x) &= \int (g(t)-\delta(t))x(t)dt = \frac{1}{2\pi}\int \overline{(\hat{g}(\omega)-1)}\hat{x}(\omega)d\omega \\ &= \frac{1}{2\pi}\int \overline{\frac{(\hat{g}(\omega)-1)}{(-j\omega)^n}}(j\omega)^n\hat{x}(\omega)d\omega = \frac{1}{2\pi}\int \overline{\hat{k}_n(\omega)}\widehat{x^{(n)}}(\omega)d\omega. \end{aligned} \tag{8}$$

The hat over a function indicates the Fourier transform of the function, and the bar over an expression indicates the complex conjugate of the expression. From this it can be seen that

$$\hat{k}_n(\omega) = \frac{\hat{g}(\omega)-1}{(-j\omega)^n}. \tag{9}$$

The conditions in (7) guarantee that the function specified in (9) remains bounded at $\omega = 0$. The above essentially constitutes a frequency-domain proof of the PKT. In this paper a filter is said to have n^{th} order accuracy if (7) is satisfied and is said to have maximum order of accuracy n if n is the largest integer for which (7) is satisfied.

B. Scaling law for Peano kernels

It is generally of interest to consider not just a single filter, $g(t)$, but a one-parameter family of filters given by

$$g_T(t) = \frac{1}{T}g\left(\frac{t}{T}\right), \text{ or equivalently } \hat{g}_T(\omega) = \hat{g}(\omega T). \tag{10}$$

The bandwidth of the filter is controlled by the scaling parameter, T, which can be adjusted to optimize performance in any particular application. The Peano kernels scale with T in a very simple way

$$\hat{k}_{l,T}(\omega) = T^l\hat{k}_l(\omega T), \text{ or equivalently } k_{l,T}(t) = T^{l-1}k_l\left(\frac{t}{T}\right). \tag{11}$$

where $k_{l,T}(t)$ is the Peano kernel of order l corresponding to $g_T(t)$. The frequency domain equation of (11) can be derived by direct substitution of (10) into (9). The time-domain equation follows directly from the frequency-domain equation.

III. Useful Relationships

A. Equivalent conditions for n^{th} order accuracy

There are conditions that are equivalent to (7) for guaranteeing that a filter has n^{th} order accuracy that are sometimes easier to apply. Equation (7) concerns the coefficients in the Taylor series of $\hat{g}(\omega)$ expanded at $\omega = 0$. Similar results are true concerning the Taylor coefficients of related functions. Let

$$\begin{aligned}\hat{g}(\omega) &= \sum_{k=0}^{\infty} a_k \omega^k,\\ \hat{h}(\omega) &= \frac{1}{\hat{g}(\omega)} = \sum_{k=0}^{\infty} b_k \omega^k, \text{ and}\\ \gamma(\omega) &= \ln(\hat{g}(\omega)) = \sum_{k=0}^{\infty} c_k \omega^k.\end{aligned} \tag{12}$$

The following three sets of conditions are then equivalent

$$\begin{aligned}&\text{A. } a_0 = 1,\ a_k = 0 \text{ for } 1 \le k \le n-1,\ a_n = \beta_n\\ &\text{B. } b_0 = 1,\ b_k = 0 \text{ for } 1 \le k \le n-1,\ b_n = -\beta_n\\ &\text{C. } c_0 = 0,\ c_k = 0 \text{ for } 1 \le k \le n-1,\ c_n = \beta_n.\end{aligned} \tag{13}$$

Any of these three conditions guarantees the filter with impulse response $g(t)$ has maximum order of accuracy n. It will later be shown that the value of a_n is important, and the result above shows that this coefficient has the same value (except for sign) for all three series.
The proof of the equivalence of the three expressions is based on the mathematical theorem that asserts that a function can be expressed as a convergent power series in only one way. This means that two power series that converge to the same function must have all of their coefficients equal. To prove the equivalence of the first two lines of (13), start with the expansion

$$\hat{g}(\omega) = 1 + a_n\omega^n + a_{n+1}\omega^{n+1} + \ldots = 1 + r(\omega). \tag{14}$$

Then

$$\hat{h}(\omega) = \frac{1}{1+r(\omega)} = 1 - r(\omega) + r(\omega)^2 + \ldots, \text{ with } r(\omega) = a_n\omega^n + \ldots \tag{15}$$

All terms of the second power of r and higher have only terms of higher power than n in ω, and the coefficient of ω^n is $-a_n$. The equivalence of the first and third lines of (13) can be established in the same manner—this time starting with the Taylor expansion of ln(1+r) about $r = 0$.

B. A simple error formula

Here it is convenient to return to the convolution formulation used in (2) and (3). Assume that g has maximum order of accuracy n. Then from (3), neglecting the noise term for the moment, and from (5) converted to convolution form we have

$$e_F(t) = (\breve{g}_T - \delta) * x(t) = \tilde{k}_{n,T}(t) * x^{(n)}(t). \tag{16}$$

Thus the error is a filtered n^{th} derivative of the input signal. If g is a low-pass filter then, by (9), so is $k_{n,T}$. The gain, G, of this filter is easily calculated as

$$G = \left|\hat{\tilde{k}}_{n,T}(0)\right| = T^n\left|\hat{\tilde{k}}_n(0)\right| = T^n\left|\hat{k}_n(0)\right| = T^n\left|\beta_n\right|, \tag{17}$$

where β_n is defined in (13). The second equality above follows from (11), and the third form the fact that a time reversal does not affect the magnitude of the Fourier transform. The derivation of the last equality follows.

$$\hat{k}_n(0) = \lim_{\omega\to 0}\frac{\hat{g}(\omega)-1}{(j\omega)^n} = \lim_{\omega\to 0}\frac{\beta_n\omega^n + \ldots}{(j\omega)^n} = j^{-n}\beta_n. \tag{18}$$

The omitted terms in the next to last expression above are of higher powers than n, so their division by ω^n go to zero as ω goes to zero. Taking absolute values gives the desired result. The simple error formula is

$$e_F(t) = \beta_n T^n \breve{x}^{(n)}(t), \text{ with } \breve{x}^{(n)}(t) \cong x^{(n)}(t), \tag{19}$$

where the second half of (19) is precisely stated as

$$\breve{x}^{(n)}(t) = K_{n,T}(t) * x^{(n)}(t), \text{ with } K_{n,T}(t) = \tilde{k}_n(t/T)\Big/\int \tilde{k}_n(t/T)dt. \tag{20}$$

The filter impulse response, $K_{n,T}(t)$, is the time reversal of the Peano kernel of order n, normalized to have a gain of one at zero frequency.

IV. Equivalent Noise Bandwidth

The aim of this paper is the estimation of the filtering error, $e_F(t)$; however, since the purpose of filtering is to reduce the noise, the effect of the filter on the noise cannot be disregarded. When comparing the filtering errors of two filters, $g_{1,T}$ and $g_{2,T}$, it is important that the parameter T associated with each filter be selected so that the two filters have the same reduction of the noise. This is accomplished by comparing filters with the same equivalent noise bandwidth [2]. For a filter with impulse response, $h(t)$, and frequency response, $\hat{h}(\omega)$, the equivalent noise bandwidth, B_{eq}, and equivalent averaging time, T_{eq}, are given by

$$B_{eq} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{h}(\omega)\right|^2 d\omega = \int_{-\infty}^{\infty} h(t)^2 dt, \quad T_{eq} = 1/B_{eq}. \tag{21}$$

The two formulas for B_{eq} are equivalent by Parseval's relation. When white noise is filtered, the root-mean-square (rms) value of the output is the same as it would be if it were filtered by an ideal low-pass filter of bandwidth B_{eq}. The rms value of the output is also the same as if the white noise were filtered by a moving average of duration T_{eq}.

For a one-parameter family of filters parameterized by the time scale, T, the equivalent noise bandwidth and averaging time only need to be calculated for the case with $T = 1$. Letting

$$B_1 = \frac{1}{2\pi}\int\left|\hat{g}(\omega)\right|^2 d\omega, \text{ and } T_1 = 1/B_1. \tag{22}$$

we have for any T

$$B_{eq} = \frac{1}{2\pi}\int\left|\hat{g}_T(\omega)\right|^2 d\omega = \frac{1}{2\pi}\int\left|\hat{g}(\omega T)\right|^2 d\omega = \frac{1}{2\pi T}\int\left|\hat{g}(\omega')\right|^2 d\omega' = B_1/T, \text{ and } T_{eq} = T_1 T. \tag{23}$$

Making the substitution $T = T_{eq}/T_1$ into (19) gives

$$e_F(t) = \left(\frac{\beta_n}{T_1^n}\right) T_{eq}^n \bar{x}^{(n)}(t) \triangleq \gamma_n T_{eq}^n \bar{x}^{(n)}(t). \tag{24}$$

V. Examples of Peano Kernels

In this section five examples of filters and their Peano kernels are presented along with a table giving the important numbers for each. Graphs of the normalized kernels are also given. All examples have symmetric impulse responses, making the distinction between g and $\tilde{g}$ unnecessary.

A. Simple averaging

The impulse and frequency responses are given by

$$g(t) = \begin{cases} 1 \text{ for } |t| \le 1/2 \\ 0 \text{ for } |t| > 1/2 \end{cases}, \quad \hat{g}(\omega) = \frac{\sin(\omega/2)}{(\omega/2)}. \tag{25}$$

The time domain formula for equivalent bandwidth gives $B_1 = T_1 = 1$. Expanding the sine function in a Taylor series gives

$$\hat{g}(\omega) = \frac{(\omega/2) - (1/6)(\omega/2)^3 + \ldots}{(\omega/2)} = 1 - \frac{1}{24}\omega^2 + \ldots. \tag{26}$$

This shows that the maximum order of accuracy is 2 and that $\beta_2 = \gamma_2 = -1/24$.

B. Symmetric Butterworth

This group of filters has a frequency response of the form

$$\hat{g}(\omega) = \frac{1}{1+\omega^n}, \text{ for even } n. \tag{27}$$

This is the magnitude squared of the frequency response, for the usual, causal, Butterworth filter. Here it is the frequency response of the filter itself, which has an impulse response that is symmetric about $t = 0$. The equivalent noise bandwidth and averaging time are given by

$$B_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(1+\omega^n)^2} d\omega = \frac{n-1}{n^2 \sin(\pi / n)}, \text{ and } T_1 = \frac{n^2 \sin(\pi / n)}{n-1}. \tag{28}$$

The integral can be evaluated using a table of integrals or a symbolic mathematics program. The special cases of $n = 2$ and $n = 4$ will be considered later. For this group of filters, the Fourier transform of the Peano kernel of maximum order has a particularly simple form:

$$\hat{k}_n(\omega) = \frac{1}{(-j\omega)^n}\left(\frac{1}{1+\omega^n} - 1\right) = \frac{1}{(-j\omega)^n}\left(\frac{\omega^n}{1+\omega^n}\right) = (-1)^{\frac{n}{2}} \frac{1}{1+\omega^n}. \tag{29}$$

In this rare case the Peano kernel is, except for sign, the same as the original filter.

C. Super Gaussian

This group of filters has a frequency response of the form

$$\hat{g}(\omega) = \exp(-\omega^n), \text{ for } n \text{ even}. \tag{30}$$

For $n = 2$ this is a Gaussian filter. From (13) C it is clear that the maximum order of accuracy is n and that $\beta_n = -1$. The equivalent noise bandwidth and averaging time are given by

$$B_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-2\omega^n) d\omega = \frac{\Gamma\left(1+\frac{1}{n}\right)}{2^{\frac{1}{n}}\pi}, \text{ and } T_1 = \frac{2^{\frac{1}{n}}\pi}{\Gamma\left(1+\frac{1}{n}\right)}. \tag{31}$$

D. Error formulas

The table below gives the formulas for the errors for the previously discussed filters.

Filter	T_{eq}	Error Formula	Error Formula
Averaging	T	$\left(T^2 / 24\right)\breve{x}^{(2)}(t)$	$0.0417T_{eq}^2\breve{x}^{(2)}(t)$
$1/(1+(\omega T)^2)$	$4T$	$T^2\breve{x}^{(2)}(t)$	$0.0625T_{eq}^2\breve{x}^{(2)}(t)$
$\exp(-(\omega T)^2)$	$5.01T$	$T^2\breve{x}^{(2)}(t)$	$0.0398T_{eq}^2\breve{x}^{(2)}(t)$
$1/(1+(\omega T)^4)$	$3.77T$	$T^4\breve{x}^{(4)}(t)$	$0.00494T_{eq}^4\breve{x}^{(4)}(t)$
$\exp(-(\omega T)^4)$	$4.12T$	$T^4\breve{x}^{(4)}(t)$	$0.00346T_{eq}^4\breve{x}^{(4)}(t)$

The equivalent averaging times were obtained from the formulas of this section. The expressions in the first column of error formulas come from (19) with the values of β_n from this section. The expressions in the second column of error formulas come from (24).

E. Graphs of normalized kernels

Figure 1 shows the Fourier transforms of the normalized kernels defined in (20). All are for filters with equivalent noise bandwidth of one. It can be seen that the Peano kernels are all low-pass filters with the maximum response at zero frequency. Thus, ignoring the filtering effect and estimating errors assuming a constant gain of one always produces an upper estimate of the filtering error. This will be further demonstrated in the examples that follow.

VI. Filter Error Estimates

One approach to estimating filtering errors, the one that will be demonstrated here, is to use a mathematical model for the signal to be measured—one for which the derivatives of the appropriate order can be calculated. Simple formulas such as those in the table above can then be used to estimate the maximum error.

To demonstrate this procedure and how well it works, consider the signal

$$x(t) = \exp\left(-0.02t^2\right). \tag{32}$$

For this signal both the second and fourth derivatives are maximum at $t = 0$, the values being 0.04 and 0.0048, respectively. These values were used in the formulas in the table to calculate the predicted

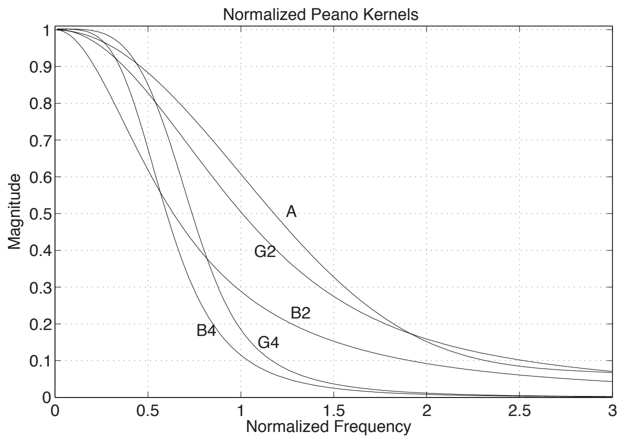

Figure 1. Graphs of the Fourier transforms of the normalized Peano kernels. All are for an equivalent noise bandwidth of 1, which has a nominal cut off frequency of 0.5. A is for the averaging filter, B for the symmetric Butterworth, and G for the super Gaussian. The numbers on the Bs and Gs represent the order of accuracy.

errors shown in Figure 2. This was done for the averaging filter and the two symmetric Butterworth filters. The agreement between the actual error and the predicted error is very good.

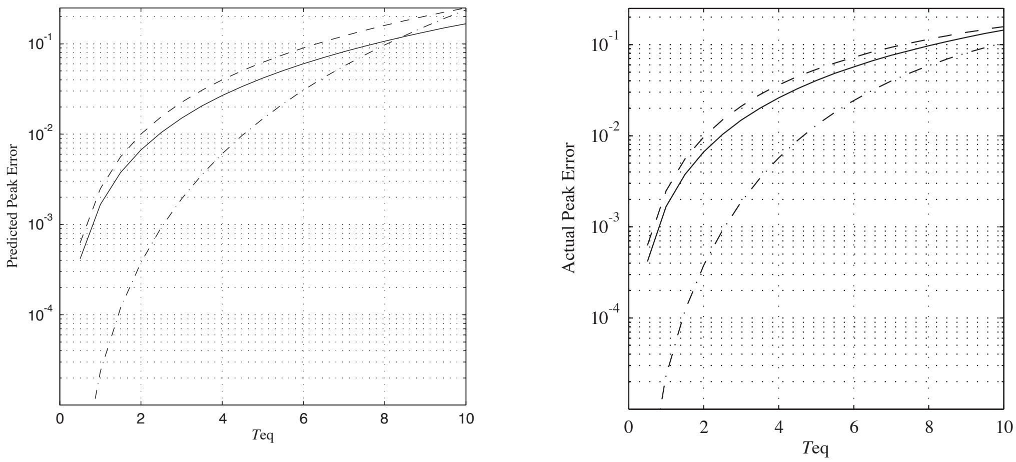

Figure 2. These plots show the predicted (from the formulas in the table above) and the actual peak errors from applying the three filters to the signal given in (32). The solid line is the averaging filter, the dashed line the 2nd order symmetric Butterworth filter. The lower curve is the fourth order symmetric Butterworth filter. The actual errors are smaller than the predicted errors for larger values of T_{eq}, because the actual error is based on the filtered second or fourth derivative while the predicted error is based on the unfiltered value. The affect is larger for the symmetric Butterworth filters than for the averaging filter because, as can be seen in Figure 1, the Butterworth kernels have lower bandwidth than the averaging kernel.

VII. Conclusion

The Peano Kernel theorem was shown to provide simple and accurate estimates for the error caused by a filter. The estimates can be calculated using straightforward algebraic operations on the filter transfer function.

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