A Maximum Entropy Alternative to the U-shaped Distribution for the Evaluation of Mismatch Uncertainty in Radio-frequency and Microwave Measurements

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Abstract – The Maximum Entropy probability density function associated with the testable information on the range and the first and second order moments of the random quantity of interest is derived. The validity of such approach is experimentally demonstrated through comparison between the proposed maximum entropy distribution and the U-shaped distribution. The underlying physical assumptions which must be satisfied for the proper use of the U-shaped distribution are revisited in the context of wide-band applications, such as Electromagnetic Compatibility measurements.

I. Introduction

Measurement uncertainty originates both from the intrinsic fluctuations of the observed quantities and from incomplete knowledge about the measurement process, this second term including, for example, the instrumentation inaccuracies. The Guide to the Evaluation of Uncertainty in Measurement (GUM) [1] first introduced a unified approach based on the use of the probability density functions (PDFs) to deal with uncertainty, no matter how uncertainty originates.

In some cases the PDF describing the distribution of the possible values attained by a quantity may be suggested by physical insight. In many other cases a physical model may be unavailable or too involved, and a PDF must be derived in some way from the available information. Associating a PDF to incomplete knowledge is a subtle and open to question process. The GUM suggests that Maximum Entropy (ME) may represent a feasible and objective criterion for deriving a candidate PDF from testable information, i.e. information expressed in terms of a range for the quantity (possibly infinite) and its moments. A detailed analytical development of the results reported in GUM and concerning ME is reported in [2].

The purpose of this work is to show that the PDF obtained from the physical model of a phenomenon can lead to a too pessimistic uncertainty estimation when the model, in order to be tractable with closed form formulas, is over-simplified. In such a case the PDF derived through a deductive physical reasoning (physical PDF) has no more value than an alternative one chosen by means of inductive reasoning from some empirical evidence, such as the range and the sample moments.

The physical PDF we will deal with in this paper is the U-shaped one. We will briefly describe its origin its application in radiofrequency and microwave measurements and the relevant simplifying assumptions in the physical model (section II). An experimental result in a representative situation is offered aimed at demonstrating the limitations of the U-shaped PDF when used to describe the distribution of data obtained from a radio-frequency measurement (section III). An alternative ME-PDF is derived through an inductive process starting from the knowledge of the range, mean value and variance of the observed random quantity (section IV). Conclusions follow.

II. The U-shaped probability density function

The U-shaped distribution is universally adopted to describe the random fluctuations of the power absorbed by a load due to the mismatch at the source side and at the load side of the cable connecting the source with the load (see [3] to [6]). The ratio between the power P_L that is actually absorbed by the load and the power P_0 that would be absorbed if the load was perfectly matched to the cable is given by

$$\frac{P_L}{P_0} = \frac{1-\left|\Gamma_L\right|^2}{\left|1-\Gamma_S\Gamma_L e^{-2\alpha l} e^{-j4\pi l/\lambda}\right|^2} \tag{1}$$

where Γ_S is the complex reflection coefficient of the source, Γ_L is the complex reflection coefficient of the load ($\left|\Gamma_S\right| \le 1$ and $\left|\Gamma_L\right| \le 1$), $e^{2\alpha l}$ is the power attenuation due to cable losses, l is the length of the cable, λ is the wavelength in the cable and j is the imaginary unit. The term $1-\left|\Gamma_L\right|^2$ represents the fraction of the transmitted power absorbed by the load and it is usually independently measured or included in the calibration of the load[1]. On the contrary the term at the denominator of equation (1), which represents the multiple reflections between the source and the load, can not be easily corrected for either by measurements or predictions and it is neglected. The amplitude and the phase of $\Gamma_S\Gamma_L e^{-2\alpha l} e^{-j4\pi l/\lambda}$ should be known at each frequency of interest, and this is not achievable in practice. The

[1] As in the case where the load is a transmitting antenna: the antenna mismatch is included in the realized gain.

fluctuations due to multiple reflections can however be statistically dealt with, in terms of maximum amplitude, mean and standard deviation.

The uncertainty associated with this approximation is evaluated as follows [4], [5]. It is assumed that:

1. Losses are negligible,
2. $|\Gamma_S \Gamma_L| << 1$.

Now, if φ is the phase of the term $\Gamma_S \Gamma_L e^{-j4\pi l/\lambda}$, then expressing the effect of multiple reflections in dB units we obtain from equation (1)

$$x = 10\log\left(\frac{P_L}{P_0\left(1-|\Gamma_L|^2\right)}\right) \approx 8.686|\Gamma_S \Gamma_L|\cos\varphi = k\cos\varphi \tag{2}$$

where $k = 8.686|\Gamma_S \Gamma_L|$. It is easy to show (see [7], pp. 97-99) that if

3. the phase φ is uniformly distributed over the interval $(0,\pi)$, then *x* follows the U-shaped distribution in the range $(-k,+k)$. The PDF of *x* is thus given by

$$f(x) = \frac{1}{\pi\sqrt{k^2 - x^2}} \tag{3}$$

when $-k < x < k$, and is zero when $x \le -k$ and $x \ge k$. The mean value of *x* is zero, and its root-mean-square value is $k/\sqrt{2}$. Note that the U-shaped distribution has a single parameter *k*. Thus once that the range for the random variable *x* is assigned, the root-mean-square value results as a consequence.

In the next section III we will demonstrate that, in practical circumstances, the U-shaped PDF is inadequate to describe the actual distribution of the log power-ratio values.

III. Mismatch in a radio-frequency and wide-band experiment

In many modern radiofrequency applications, the generation and measurement of electrical quantities are performed over wide frequency ranges. For example, Electromagnetic Compatibility radiated emission tests for commercial equipment cover the (30-1000) MHz frequency range. Long cables (about 10 m or more, i.e. several wavelengths) are used to connect the antenna to a relatively distant receiver. Thus cable losses are not negligible. Also, broadband antennas are used in order to cover the whole frequency span with a single antenna. The mismatch of broadband antennas is relatively large, then the product $|\Gamma_S \Gamma_L|$ is not much less then unity ($|\Gamma_S \Gamma_L|$ values of the order of 0.2 can be found in practice).

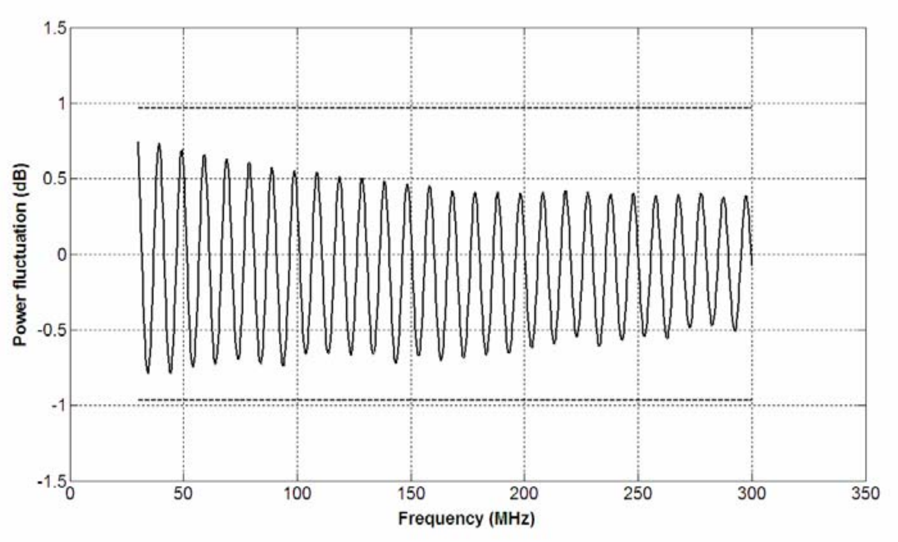

Figure 1: fluctuations of the measured power due to source and load mismatch (continuous line). Dashed lines correspond to the upper and lower theoretical limiting values as given by equation (2) for $\varphi = 0$ and $\varphi = \pi$.

A section of RG 58 C/U type cable (50 Ω nominal characteristic impedance) of 10 m length is connected between a mismatched source and a mismatched load. The resistance of the source is deliberately set to 25 Ω by placing a 50 Ω feedthrough termination at the output of the 50 Ω generator feeding the cable. A 50 Ω feedthrough termination is also connected at the input of a spectrum analyzer and the feedthrough-spectrum analyzer combination loads the cable, thus realizing a 25 Ω load. The reflection coefficients at the source and at the load are equal, essentially frequency independent and given by $\Gamma_S = \Gamma_L = -1/3$. The generator sweeps the 30-300 MHz frequency range, then the term $e^{-j4\pi l/\lambda}$ completes about 27 turns around the origin of the complex plane. The spectrum analyzer measures the power absorbed by the load and the observed fluctuation is plotted in figure 1. The oscillatory waveform is originated from the multiple reflections along the cable and the damping of the oscillation is caused by the cable losses. The vertical scale is in dB units for direct comparison with equation (2). The straight continuous lines represents the extreme values $\pm k = \pm 0.965$ dB, as predicted by equation (2) when $\varphi = 0$ and $\varphi = \pi$, respectively. The mean value of the oscillation is -0.097 dB, the peak value is 0.745 dB and the standard deviation is 0.406 dB. Note that the peak value of the oscillation and its standard deviation are much less than the predicted ones, i.e. $k = 0.965$ dB and $k/\sqrt{2} = 0.682$ dB, respectively. This is due to cable losses: intermediate values of the oscillation amplitude are more frequent than it would be expected if damping was negligible.

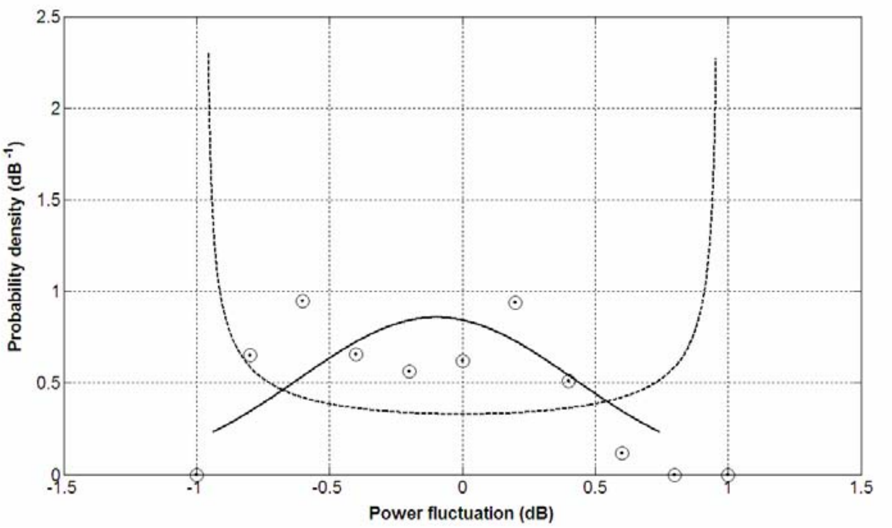

Figure 2: comparison between the U-shaped PDF (dashed line) computed according to equation (3) with $k = 0.965$ dB, the ME-PDF (continuous line) given by (6) with $\lambda_0 = 0$ and $k = 0.858$ dB^{-1}, $\lambda_1 = -1.851$ dB^{-2}, $\mu = -0.097$ dB, and the observed PDF (circles with dots) as obtained from the measured data in figure 1. The U-shaped distribution is zero outside the interval $(-0.965, +0.965)$ dB, while the ME-PDF is zero outside the interval $(-0.940, +0.745)$ dB.

In figure 2 is represented the U-shaped distribution (dashed line) as obtained from equation (3), where *k* equals the predicted value 0.965 dB. The bell-shaped curve (continuous line) is the ME-PDF, derived as described in the next section IV, whose parameters are calculated from the mean value, the range and the standard deviation of the measured data. The circles with dots represent the histogram obtained from the measured data and expressed in terms of relative frequency of occurrence of data in each bin (10 bins in the – 1 dB to + 1 dB range) divided by the bins' amplitude (0.2 dB). From figure 2 it results that the ME-PDF is more suited to quantitatively describe the distribution of the measured data than the U-shaped PDF.

IV. Maximum Entropy probability density function

If the range and the moments of a random variable (RV) are available from analysis and/or empirical evidence a PDF can be assigned to the RV by resorting to the ME criterion. The criterion consists in choosing the PDF whose entropy is maximum given some constraints. The constraints must consist of testable information, in practice the range and the moments of the RV. The entropy *S* associated with the PDF *f*, function of the RV *x*, is defined as:

$$S = -\int f(x) \ln\left[\frac{f(x)}{m(x)}\right] dx \tag{4}$$

where the measure $m(x)$ assures that the entropy is invariant with respect to a change of variable. A uniform measure is postulated, corresponding to complete ignorance about the value of *x*, and its explicit expression (a constant) will be omitted in the following derivations.

We assume that the mean μ and the variance σ^2 of the RV *x* are known or estimated, and that the possible values of *x* are confined in a symmetric interval around μ whose half-amplitude is a. The process by which the ME-PDF is derived is known as "constrained optimization" and is usually dealt with the method of the Lagrange multipliers. The method consists in determining the PDF $f(x)$ which maximizes expression (5), where the first term on the right side is the entropy defined by equation (4):

$$Q = -\int_{\mu-a}^{\mu+a} f(x)\ln\left[f(x)\right]dx + \lambda_0\left[\int_{\mu-a}^{\mu+a} x\, f(x)\,dx - \mu\right] + \lambda_1\left[\int_{\mu-a}^{\mu+a}\left(x-\mu\right)^2 f(x)\,dx - \sigma^2\right] \tag{5}$$

By solving for f the equation $\dfrac{dQ}{df} = 0$, we obtain

$$f(x) = k\exp\left[\lambda_0\left(x-\mu\right) + \lambda_1\left(x-\mu\right)^2\right] \tag{6}$$

Now, the values of the parameters k, λ_0 and λ_1 are derived from the following constraints

$$\int_{\mu-a}^{\mu+a} f(x)\,dx = 1 \qquad \text{(normalization)} \tag{7}$$

$$\int_{\mu-a}^{\mu+a} x\, f(x)\,dx = \mu \qquad \text{(known mean)} \tag{8}$$

$$\int_{\mu-a}^{\mu+a} \left(x-\mu\right)^2 f(x)\,dx = \sigma^2 \qquad \text{(known variance)} \tag{9}$$

Substituting equation (6) into equations (7), (8) and (9) we obtain $\lambda_0 = 0$ and

$$\begin{cases} k = j\sqrt{\dfrac{\pi}{\lambda_1}}\dfrac{1}{\exp\left(\lambda_1 a^2\right)w\left(\sqrt{\lambda_1}a\right)-1} \\ -\dfrac{1}{2\lambda_1} + j\dfrac{a}{\sqrt{\pi\lambda_1}}\dfrac{\exp\left(\lambda_1 a^2\right)}{\exp\left(\lambda_1 a^2\right)w\left(\sqrt{\lambda_1}a\right)-1} = \sigma^2 \end{cases} \tag{10}$$

where $w(z)$ is the complex error function. The solution of the non-linear system of equations (10) cannot be obtained in closed-form and it is necessary to resort to numerical computation. First the second equation of (10) is solved for λ_1 through an iterative numerical procedure, then k is calculated from the first equation. It can be proved, by rearranging equations (10), that the normalized parameters $k\sigma$ and $\lambda_1\sigma^2$ are functions of the ratio a/σ ($1 < a/\sigma < \infty$) and not of a and σ separately. Thus the solution of the system of equations (10), when expressed in terms of the normalized parameters can be graphically represented by the universal plots reported in figure 3 and in figure 4.

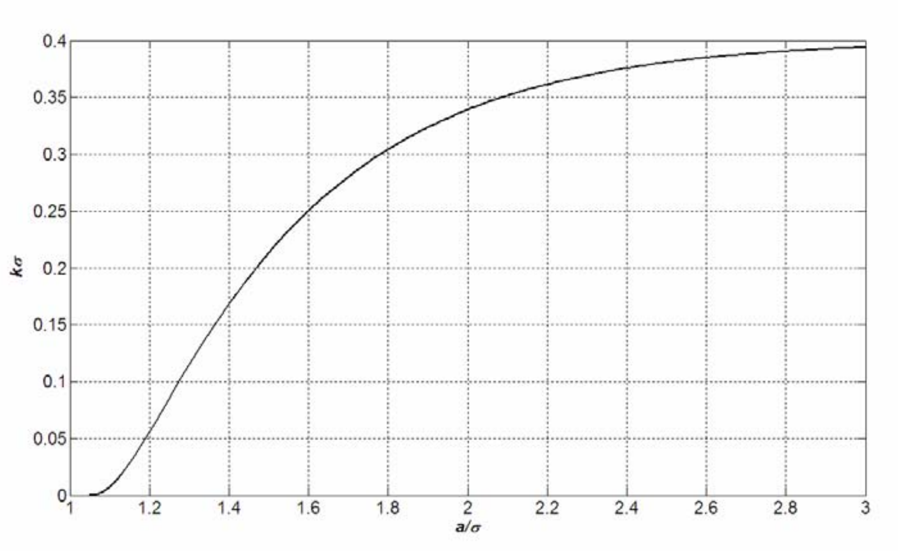

Figure 3: normalized parameter $k\sigma$ as a function of the a/σ ratio.

It is interesting to observe that when a/σ tends to infinity then k tends to $1/\left(\sqrt{2\pi}\sigma\right)$, and λ_1 tends to $-1/\left(2\sigma^2\right)$, thus $f(x)$ tends to the normal distribution. When $a/\sigma=\sqrt{3}$ we have $k=1/(2a)$ and $\lambda_1=0$, thus $f(x)$ is uniform. When $a/\sigma>\sqrt{3}$ we have $k>1/\left(\sqrt{2\pi}\sigma\right)$ and $\lambda_1<0$, thus $f(x)$ is concave. On the contrary when $1<a/\sigma<\sqrt{3}$ we have $0<k<1/\left(\sqrt{2\pi}\sigma\right)$ and $\lambda_1>0$, thus $f(x)$ is convex.

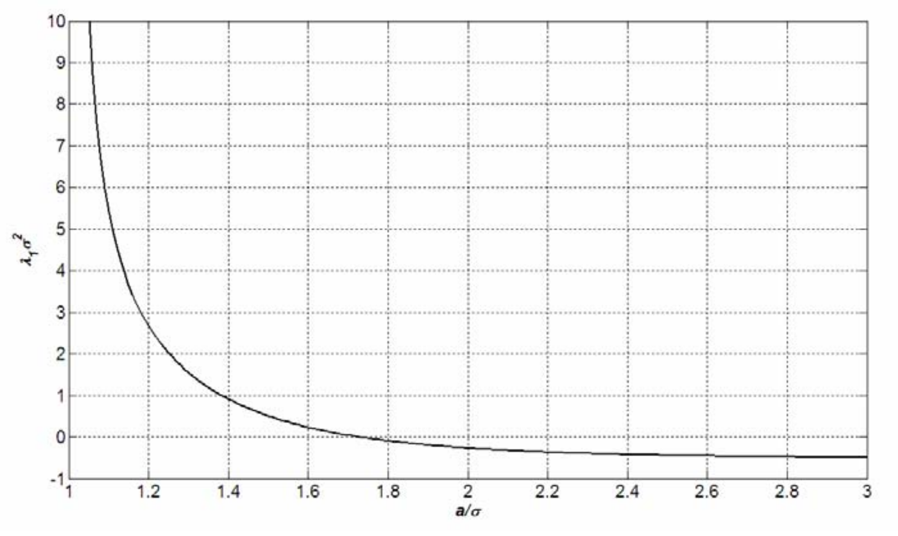

Figure 4: normalized parameter $\lambda_1\sigma^2$ as a function of the a/σ ratio.

Recall that if x is U-shaped distributed over an interval whose half-amplitude is a we have $\sigma=a/\sqrt{2}$, i.e. $a/\sigma=\sqrt{2}$. Now, the ME-PDF having the same range and variance has parameters $k\approx 0.1753/\sigma$ and $\lambda_1\approx 0.8460/\sigma^2$. For comparison in figure 5 are plotted both the U-shaped PDF (continuous line) and the corresponding ME-PDF (dashed line), in the particular case where $a=1$, $\sigma=1/\sqrt{2}$ and $\mu=0$.

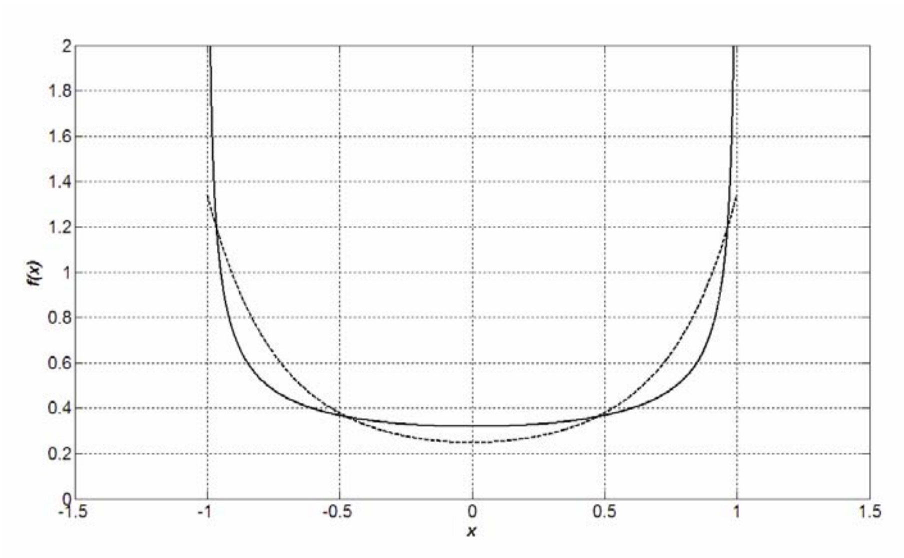

Figure 5: comparison between the U-shaped distribution (continuous line), and the corresponding ME-PDF with $a/\sigma = \sqrt{2}$ (dashed line). Specific case corresponding to $a = 1$, $\sigma = 1/\sqrt{2}$ and $\mu = 0$.

V. Conclusions

In this work the problem of the determination of the Maximum Entropy probability density function in the case where the testable information consists of the range, mean and variance of the random variable is solved. Although the solution is worked out through numerical computation, it is expressed through universal plots, valid for any specific value of the range, mean and variance. It is demonstrated that neglecting cable losses in a wide-band radiofrequency transmission experiment or test, as it is implicitly done when adopting the U-shaped distribution, can lead to severely overestimating the mismatch uncertainty. The proposed Maximum Entropy alternative better fits the distribution of the random values experimentally originating from multiple reflections along a transmission line when cable losses cannot be neglected. An experimental confirmation is offered.

References

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