To the uncertainty of DFT-based DSP algorithms used for processing multifrequency signals

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***Abstract*-**The paper analyses effect of windowing a multi-frequency signal on the result of calculating signal parameters based on DFT. Influence of window side-lobes of nearby harmonic signals on the result of computation depends on the on the used window and on the difference in the two components magnitudes and frequencies. There are several other sources of resulting bias and uncertainty of measurements based on such algorithms. These influences are also briefly mentioned. The presented examples concern two methods of estimating RMS value, but the described principle is applicable to all algorithms based on DFT. Both theoretical relations and simulation results are included.

I. Introduction

When analyzing properties of various algorithms used in measurement, the analysis is often oriented primarily on sinusoidal signal, since such a signal presents the basic element of periodic and multi-frequency signals. If there are more harmonic and/or inter-harmonic components in the signal to be processed, some additional sources of measurement uncertainty may have to be taken into account. Frequency spectrum of windowed multi-frequency signal is a superposition of several window spectra. The number of these spectra is equal to double of the number of signal harmonic components (sinusoidal components with various phase shifts). Levels of the individual spectra correspond to (halves of) magnitudes of signal harmonic components, and frequencies of individual window spectra maximums are equal to positive and negative frequencies of harmonic spectral components of the measured signal. This is caused by the fact, that sinusoidal signal is composed of two complex exponential signals (with imaginary exponents) with positive and negative sinusoidal signal frequency. We suppose that the used window is originally real and even function. Its spectrum in this case is also real and even. Even window is not causal. Since data are usually processed for positive time values, the window is before applying to signal shifted by half of its length. That shifting in time domain results in linear phase of the window spectrum. The corresponding window group delay is (N/2).*T* if the window length is N and sampling interval is *T*.. Since periodic extension of the sampled part of signal is supposed in algorithms based on DFT, the last sample of the originally even window is removed before processing. (This type of window is called 'periodic' in the last versions of MATLAB.)

Multiplication of window spectra by the two exponential functions representing a harmonic component of the processed multi-frequency signal corresponds in the frequency domain shifting the window spectrum (without changing amplitude or phase spectrum) to two frequencies equal to the sinusoidal signal frequency with positive and negative sign. The amplitude (module) spectra are even and the phase spectra are odd. The two phase spectra are linear and their slopes are preserved by the frequency domain shift. The phase slopes have different signs, which corresponds the fact that the phase spectrum of the real window should be odd.

If the signal sampling is coherent (i.e. if the sampled part of the harmonic component represents an integer number of signal periods, no leakage occurs. If the well-known cosine windows are used, each of the two spectra corresponding to each harmonic component of the multi-frequency signal is composed of only of the 2P+1 spectrum lines placed on the window main lobe on the DFT grid. If the sampling is non-coherent, which is a frequent case in practice, leakage occurs and the DFT window spectrum side-lobe components are nonzero. These components of each shifted window spectrum can be non-negligible in the frequency area of the spectrum main-lobe of another multi-frequency signal part. (A bias of the RMS value of each signal component may occur in this case.) The influence of these window spectrum side-lobes is investigated in this paper and for the case of a sinusoidal signal it is demonstrated in Fig.1.

II. Accuracy of RMS value estimate based on DFT – what is it influenced by

Measurement of RMS value can be performed by processing the input samples sequence either in time domain, or in frequency domain. The frequency domain algorithms are based on DFT. There are two basic algorithms used for RMS estimation in frequency domain – processing only the windowed signal DFT spectrum components inside the main lobe of the used window spectrum [1-3], or interpolation of DFT spectrum of the windowed signal [3 - 7].
The accuracy of RMS estimate is in both algorithms influenced by uncertainty of signal samples, caused mainly by quantization noise. A contribution to this uncertainty by an external noise is also possible. The variance (square of standard uncertainty) of the RMS value can be found by the uncertainty propagation law and it is [3]

$$u^2(X_{RMS}) = u_n^2 \frac{1}{N^2 \times nnpg^2} \sum_{n=0}^{N-1} w^4 = \frac{u_n^2 \times ENBW_0}{N} \tag{1}$$

where u_n is standard deviation of quantization or another external noise affecting input samples, N is DFT length, *nnpg* is normalized noise power gain and $ENBW_0$ is equivalent-noise bandwidth of the squared window

$$ENBW_0 = N \frac{\sum_{n=0}^{N-1} w^4(n)}{\left(\sum_{n=0}^{N-1} w^2(n)\right)^2} \tag{2}$$

A negative bias of RMS estimation by the "main lobe algorithm" is caused by ignoring the DFT spectrum outside the main lobe. The worst-case relative bias was also analyzed in [3] and can be found for the most frequently used cosine windows of the order P and with coefficients a_P as

$$\delta_{RMS} = \frac{X'_{RMS} - X_{RMS}}{X_{RMS}} = \frac{1}{\pi\sqrt{nnpg}} \sqrt{2\sum_{i=0}^{P-1}\left(\sum_{p=0}^{P} a_p \frac{(0,5+i)}{(0,5+i)^2 + p^2}\right)^2 + \left(\sum_{p=0}^{P} a_p \frac{(0,5+P)}{(0,5+P)^2 + p^2}\right)^2} - 1 \tag{3}$$

Another component of RMS bias that may have to be taken into account for some DFT spectra shapes is the positive bias caused by parts of a large adjacent spectral components falling into the main lobe frequency band of a week spectral component. This issue was not analyzed in [3] and *its evaluation for the both above-mentioned methods of RMS value estimation in the frequency domain is the main goal of this paper*.

III. Bias of RMS value found from window spectrum main-lobe caused by window side-lobes

A description of this method can be found e.g. in [1- 3].
The RMS value of a sine wave, X_{RMS}, can be computed from the positive and negative frequency bands of $2P+1$ FFT amplitude spectrum components centered at the frequency of interest corresponding to index k of the spectrum component [1] that can be found e.g. as the local maximum in the amplitude frequency spectrum. The signal RMS value is

$$X_{RMS} = \sqrt{\frac{1}{N^2 nnpg}\left(\sum_{i=k-P}^{k+P} M^2(i) + \sum_{j=-k-P}^{-k+P} M^2(j)\right)} \tag{4}$$

where N is DFT length, *nnpg* is normalized noise power gain, and M(i) is the i-th component of the two-sided signal frequency spectrum . P is used cosine window order.

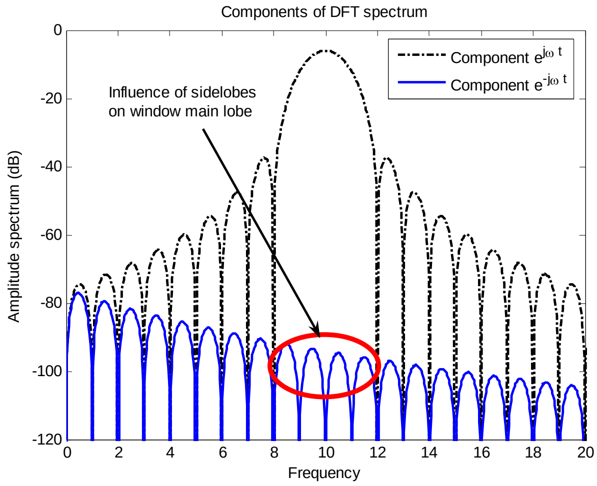

Figure 1. Demonstration of window side-lobes of the mirrored windowed component influencing the main-lobe spectrum part of the investigated windowed component

The formula (1) can be for real signals rewritten as

$$X_{RMS} = \sqrt{\frac{2}{N^2 nnpg}\left(\sum_{i=k-P}^{k+P} M^2(i)\right)} \tag{5}$$

where M(i) is the i-th component of the one-sided signal frequency spectrum. Spectral lines corresponding to the side-lobes of remaining signal spectrum components including the mirrored spectral component are added to all spectral lines of each signal spectrum line. (Additive noise of signal is not considered here, it can be evaluated using (1).)
The formula (4) can therefore be rewritten in the form taking into account the influence of spectrum side-lobes of other spectral components

$$X_{RMS} = \sqrt{\frac{2}{N^2 nnpg}\left(\sum_{i=k-P}^{k+P} \left|X_{harm}(i) + X_{sidelobes}(i)\right|^2\right)} \tag{6}$$

Here X_{harm} is the spectrum corresponding to one complex exponential function (one of two parts of a harmonic component) and $X_{sidelobes}$ is the spectrum without X_{harm} (in practice a part of side-lobe spectrum of a nearby dominant spectral component, placed in the frequency area corresponding to the spectrum main-lobe of X_{harm}). Complex spectrum components are added here, so the result of the sum depends on spectral components phases. Phase differences of the two spectrum components in (6) depend primarily on the used window. If the spectral components to be added are mutually perpendicular, we can write

$$X_{RMS} = \sqrt{\frac{2}{N^2 nnpg}\left(\sum_{i=k-P}^{k+P} M_{harm}^2(i) + M_{sidelobes}^2(i)\right)} = \sqrt{X_{RMSact}^2 + X_{RMSsidelobes}^2} \tag{7}$$

Here $M_{harm} = | X_{harm} |$ are values of amplitude (or module) spectrum, and $X_{RMSsidelobes}$ is the *RMS* value corresponding to the side-lobes of other harmonic components in the main-lobe of the evaluated harmonic component:

$$X_{RMSsidelobes} = \sqrt{\frac{2}{N^2 nnpg}\left(\sum_{i=k-P}^{k+P} M_{sidelobes}^2(i)\right)} \tag{8}$$

The side-lobes can be estimated from the used window spectrum. It is very simple for Hamming window, where the side-lobes fall-off is practically negligible. If no leakage bounds of individual spectrum components are known, the worst case should be inspected. For finding a rough bias estimate, one value of the side-lobe component module can be used for the whole window spectrum main-lobe area of the estimated harmonic component $M_{SL} \cong M_{sidelobes}(i)$. In this case

$$X_{RMSsidelobes} = \sqrt{\frac{2}{N^2 nnpg}(2P+1)M_{SL}^2} \quad (9)$$

The estimate of the RMS bias component caused by side-lobes is

$$X_{RMS} - X_{RMSact} = \sqrt{X_{RMSact}^2 + X_{RMSsidelobes}^2} - X_{RMSact} \quad (10)$$

Here X_{RMSact} is the actual („true“) RMS value. The relative bias caused by window side-lobes can therefore be expressed as

$$\frac{X_{RMS} - X_{RMSact}}{X_{RMSact}} = \sqrt{1 + \frac{X_{RMSsidelobes}^2}{X_{RMSact}^2}} - 1 \quad (11)$$

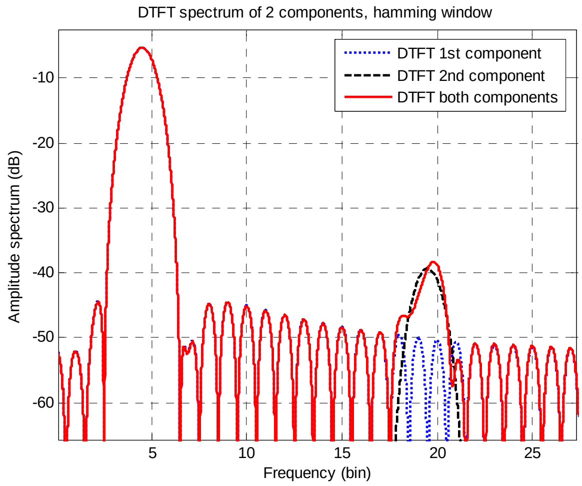

Figure 2. DTFT spectra of the 1st and the 2nd components and the resulting DTFT spectrum, Hamming window. The DTFT spectrum of the 1st component overlaps mostly with that of both components

Table 1. Two components of relative RMS bias explained in the 1st table column

window	RV1-1	RV1-2	RV1-3	RV1-4	Hamming	BH4
RMS bias caused by main-lobe components effected by close components side-lobes (%)	1.326	0.0052	0.00023	0.00047	71.41	0.0054
RMS bias caused by ignoring the investigated tone side-lobes (%)	-0.021	-0.0006	-0.000025	0.000000	-0.011	0.00000

Tab.1 presents an example of relative RMS bias components caused by spectrum side-lobes of a nearby spectral component (row 1) and by ignoring side-lobe contribution of the investigated spectral component (row 2) for the indicated windows (RV1 for Rife-Vincent of the 1st class) for the RMS estimation based on window main-lobe spectrum processing [1]. The two spectral components magnitudes were 1 V and 0.0025 V, their frequencies in bin were 10.55 and 21.10. The weak component was below side-lobes level by Hamming window, therefore the large bias in the table first row for Hamming window.

IV. Bias of RMS value found by DFT interpolated in the frequency domain caused by window side-lobes

The second method of finding RMS value of non-coherently sampled signal by processing it in the frequency domain is based on signal windowing and interpolation of windowed signal spectrum in frequency domain. If non-integer number periods of a sinusoidal signal is sampled, the signal spectral line does not lie on one of the DFT grid lines. This results in spectral leakage and there are many spectral lines instead of one for positive and negative frequencies in the DFT spectrum. The interpolated DFT algorithms find the decimal frequency δ (that must be added (with proper sign) to the local maximum DFT spectral line frequency to get the actual signal frequency) from ratios of the lengths of local maximum amplitude spectrum line and its two neighboring lines [4-7]. Knowing actual frequency, the used window shape and the length of the longest DFT spectrum component allows finding the actual sinusoidal component magnitude and phase, that means also the investigated signal component RMS. In the case of the multi-frequency signal, local maxima of windowed DFT spectrum are found for each harmonic component and the described procedure is applied to each of the spectrum component. The Rife-Vincent windows of the first class (abbreviated RV1) [8, 5] form a subset of cosine windows allowing finding magnitude of the investigated sinusoidal signal without using iterations. These windows (of the order 1 to 4) were used in computer simulations here for the both methods of RMS estimation for the same two- frequency signal. The values of bias found are shown in Table 2.

Table 2. Relative RMS bias using windows RV1 and methods described in parts III and IV

window	RV1-1 (Hann)	RV1-2	RV1-3	RV1-4
RMS bias caused by *main-lobe* components affected by close components side-lobes (%)	1.32	0.0052	0.00023	0.00047
RMS bias caused by *IDFT*-based method affected by close components side-lobes (%)	0.042	0.042	0.0041	0.000084

Similarly like in finding RMS by processing all the DFT components in the window spectrum main lobe (part III.), here also DFT components in the window main lobe are processed, but only three of them regardless of the window order were used. (More complicated interpolations are reported in [7].) The window side lobes of another spectral component can influence the lengths of the spectral lines of the investigated component used for finding δ also in this case, if the closest harmonic component of the investigated component is not distant enough from it. The neighboring components influence (spectral interference) depends on the number of signal periods sampled, on the difference of magnitudes of the closest components and on the used window (on its amplitude and phase spectrum). If there are not very high demands on accuracy, it is sufficient, if the used window main lobes of the two adjacent spectrum lines do not overlap.

Further we present some theoretical expressions concerning estimation of basic parameters required for using interpolated DFT for signal component RMS.

Finding decimal frequency bin δ is based on ratios of the neighboring DFT windowed spectrum components in local spectrum maximum:

$$\hat{\beta}_1 = \frac{|X(k+1)|}{|X(k)|},\ \hat{\beta}_2 = \frac{|X(k-1)|}{|X(k)|} \tag{12}$$

The bias of β_1 and β_2 estimates can be found as:

$$b(\beta_i) = \frac{|X(k\pm 1)|}{|X(k)|} - \frac{|X(k\pm 1) - X_S(k\pm 1)|}{|X(k) - X_S(k)|}, i = 1,2 \tag{13}$$

Where X_S is spectrum with removed main component. The result of bias estimates in (13) depends on phase differences of X and X_S components. The worst case corresponds to the phase difference zero or π. Since the phase values of DFT spectrum (i.e. sampled continuous DTFT spectrum) change by π between two frequency bins adjacent to signal frequency bin k [9], the worst case of $b(\beta)$ is encountered, if the side-lobes phase is in-phase or opposite phase with the main component phase. The bias can be estimated from the module spectrum as

$$\left|\hat{b}(\beta_1)\right| \approx \left(\hat{\beta}_1 + 1\right)\frac{\overline{M}_S}{M(k)} \tag{14}$$

$M_S(k) \cong M_S(k+1) \equiv \overline{M}_S$ and $M(k) >> \overline{M}_S$, where M is the main component spectrum and M_S is spectrum without the main component contribution. The actual signal frequency is $f=(k+\delta).\Delta f$, $.\Delta f$

being the frequency bin and δ is the decimal frequency in bin. Bias of frequency estimation depends on bias of decimal frequency bin estimation.
The estimation of the decimal frequency bin can be found for RV1 windows as

$$\hat{\delta}_1 = \frac{(P+1)\hat{\beta}_1 - P}{(1+\hat{\beta}_1)} \tag{16}$$

The spectral component magnitude can be estimated as

$$A_1 = \frac{2\pi \cdot M(k)}{N} \frac{1}{|D_1(\delta)|}, \qquad D_1(\delta) = \gamma_0(\delta) \cdot \delta \cdot \sin(\pi\delta) \tag{17}$$

Its relative bias (shown for indicated windows in table 2) can be estimated as

$$\frac{b(A_1)}{A_1} = \frac{\hat{A}_1 - A_1}{A_1} \cong \frac{|D_1(\delta)|}{|\hat{D}_1(\hat{\delta})|} - 1 = \frac{|\hat{D}_1(\hat{\delta} - b(\delta))|}{|\hat{D}_1(\hat{\delta})|} - 1 \tag{18}$$

IV. Conclusion

The influence of window spectrum side-lobes of a nearby component on the RMS value estimation based on either window main-lobe spectral components processing or on using the interpolated DFT spectral component estimation was briefly explained and demonstrated for some frequently used windows. This procedure can be used also for multi-frequency signals, where all signal spectrum components corresponding to windowed signal amplitude spectrum local maxima have to be processed in the described way. In this case either only the closest adjacent component side-lobes or side-lobes of several components must be taken into account, depending on the used window and on amplitude and frequency difference of the investigated and other spectral components.

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