Uncertainty in Measuring the Power Spectrum Density of a Random Signal

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Abstract- The method to assess uncertainty in measuring the power spectrum density of a random waveform is presented. Then, the evaluation of assessing uncertainty of power spectrum density is carried out. It takes into account propagation of uncertainties associated with each sample recording in the algorithm of digital signal processing and an error of the estimator bias resulting from a mathematical model of the measured value. The dependences for uncertainties in measuring the power spectrum density of a random waveform are introduced.

I. Introduction

Power spectrum density calculated on the basis of the samples of the analysed signal is estimated according to the algorithm of digital signal processing (DSP). The evaluation of uncertainty in measuring spectrum density of a random waveform takes into account uncertainty associated with samples recording, propagation of this uncertainty in the DSP algorithm and a bias error of the accepted estimator. Uncertainty associated with recording of each single sample of the examined signal is affected by parameters of the measurement system hardware (among others: a/d converter resolution). When the dependence for the estimator bias of power spectrum density is expressed analytically, it is possible to take into account the influence of an error of the estimator bias in the form of uncertainty resulting from this error. Then, uncertainty in measuring power spectrum density $u(\hat{G}_n(k))$ can be assessed on the basis of the following equation:

$$u(\hat{G}_n(k)) = \sqrt{u_c^2(\hat{G}_n(k)) + u_b^2(\hat{G}_n(k))} \tag{1}$$

where $u_c(\hat{G}_n(k))$ is combined uncertainty in measuring power spectrum density, $u_b(\hat{G}_n(k))$ is uncertainty resulting from the estimator bias.

II. Uncertainty propagation in the DSP algorithm

The estimator of power spectrum density $\hat{G}_n(k)$ was calculated with the use of a Fourier transformation, where the procedure of averaging the spectrum estimator is made at time intervals (section averaging). The section averaging means that for a sampling period Δt, the time series of the analysed signal $x(i\Delta t)$ is divided into Q time intervals with the K length, which is chosen according to the selected resolution of the analysis. For each of q sections the estimators of power spectrum density are calculated with the use of the following equation [1]:

$$\hat{G}_n(k,q) = \frac{2\Delta t}{K}\left|X_k\right|^2 \tag{2}$$

where X_k are components (ordinates) of DFT transform, Δt is a sampling period, $K = N/Q$ (N -the number of samples of the analysed time waveform, Q – the number of averagings).

Next, for each frequency $f_k = \dfrac{k}{K\Delta t}$, these Q estimators of power spectral density were averaged:

$$\hat{G}_n(k) = \frac{\hat{G}_n(k,1) + .. + \hat{G}_n(k,q) + .. + \hat{G}_n(k,Q)}{Q} \tag{3}$$

In the considerations it was assumed that Δt = const. Then, combined uncertainty $u_c(\hat{G}_n(k))$ in measuring power spectrum density, with the assumption that the estimators of power spectrum density for the successive time sections are not correlated with one another, is described by the following equation [2]:

$$u_c(\hat{G}_n(k)) = \sqrt{\sum_{q=1}^{Q}(\frac{\partial \hat{G}_n(k)}{\partial \hat{G}_n(k,q)})^2 u^2(\hat{G}_n(k,q))} = \sqrt{\frac{1}{Q^2}\sum_{q=1}^{Q} u^2(\hat{G}_n(k,q))} \quad (4)$$

where $u(\hat{G}_n(k,q))$ is partial uncertainty in measuring power spectrum density for the q-th time interval.

The use of the smoothing operation for the estimator of power spectrum density decreases uncertainty in measuring power spectrum density $\sqrt{Q}$ - times with the assumption that partial uncertainties $u(\hat{G}_n(k,q))$ in measuring power spectrum density for each q-th time interval are identical.

The estimators of power spectrum density $\hat{G}_n(k,q)$ are calculated on the basis of the DFT transform of the signal $x(i\Delta t)$:

$$X_k = \sum_{i=0}^{K-1} x(i\Delta t)\exp(-j\frac{2\pi ki}{K}) \quad (5)$$

which possesses two non-correlated with each other parts: the real one *ReX(k)* and the imaginary one *ImX(k)*:

$$\begin{aligned} \operatorname{Re} X(k) &= \sum_{i=0}^{K-1} x(i\Delta t)\cos(\frac{2\pi ki}{K}) \\ \operatorname{Im} X(k) &= \sum_{i=0}^{K-1} x(i\Delta t)\sin(\frac{2\pi ki}{K}) \end{aligned} \quad (6)$$

Therefore, the dependence (2) can be written in the following way:

$$\tilde{G}_n(k,q) = \frac{2\Delta t}{K}\{[\operatorname{Re} X(k)]^2 + [\operatorname{Im} X(k)]^2\} \quad (7)$$

Partial uncertainty in measuring power spectrum density for the q-th time interval can be thus assessed on the basis of the expression:

$$u^2(\hat{G}_n(k,q)) = (\frac{\partial \hat{G}_n(k,q)}{\partial \operatorname{Re} X(k)})^2 u^2(\operatorname{Re} X(k)) + (\frac{\partial \hat{G}_n(k,q)}{\partial \operatorname{Im}(k)})^2 u^2(\operatorname{Im} X(k)) \quad (8)$$

where *u(ReX(k), u(ImX(k))* are partial uncertainties in calculating the DFT real and imaginary parts respectively.

Determining sensitivity coefficients:

$$\frac{\partial \hat{G}_n(k,q)}{\partial \operatorname{Re} X(k)} = \frac{4\Delta t}{K}\operatorname{Re} X(k), \qquad \frac{\partial \hat{G}_n(k,q)}{\partial \operatorname{Im}(k)} = \frac{4\Delta t}{K}\operatorname{Im} X(k) \quad (9)$$

as well as partial uncertainties [3]:

$$\begin{aligned} u^2(\operatorname{Re} X(k)) &= \sum_{i=0}^{K-1}(\frac{\delta \operatorname{Re} X(k)}{\delta x(i\Delta t)})^2 u^2_{x(i\Delta t)} = \sum_{i=0}^{K-1}\cos^2(\frac{2\pi ki}{K})u^2_{x(i\Delta t)} \\ u^2(\operatorname{Im} X(k)) &= \sum_{i=0}^{K-1}(\frac{\delta \operatorname{Im} X(k)}{\delta x(i\Delta t)})^2 u^2_{x(i\Delta t)} = \sum_{i=0}^{K-1}\sin^2(\frac{2\pi ki}{K})u^2_{x(i\Delta t)} \end{aligned} \quad (10)$$

where $u_{x(i\Delta t)}$ is uncertainty associated with single sample recording, whereas [3]:

$$\sum_{i=0}^{K-1} \cos^2\left(\frac{2\pi ki}{K}\right) u_{x(i\Delta t)}^2 = K$$
$$\sum_{i=0}^{K-1} \sin^2\left(\frac{2\pi ki}{K}\right) u_{x(i\Delta t)}^2 = 0 \quad \text{for } k = 0 \qquad (11)$$

and

$$\sum_{i=0}^{K-1} \cos^2\left(\frac{2\pi ki}{K}\right) u_{x(i\Delta t)}^2 = \sum_{i=0}^{K-1} \sin^2\left(\frac{2\pi ki}{K}\right) u_{x(i\Delta t)}^2 = \frac{K}{2} \quad \text{for } k \neq 0 \qquad (12)$$

combined uncertainty in measuring power spectrum density $u_c(\hat{G}_n(k))$ is expressed by the following equations:

$$u_c(\hat{G}_n(k)) = 2u_{x(i\Delta t)}\sqrt{\frac{2\Delta t}{Q}\hat{G}_n(k)} = 2u_{x(i\Delta t)}\sqrt{2\frac{\hat{G}_n(k)}{B_e N}} \quad \text{for } k = 0 \quad (13)$$

$$u_c(\hat{G}_n(k)) = 2u_{x(i\Delta t)}\sqrt{\frac{\Delta t}{Q}\hat{G}_n(k)} = 2u_{x(i\Delta t)}\sqrt{\frac{\hat{G}_n(k)}{B_e N}} \quad \text{for } k \neq 0 \quad (14)$$

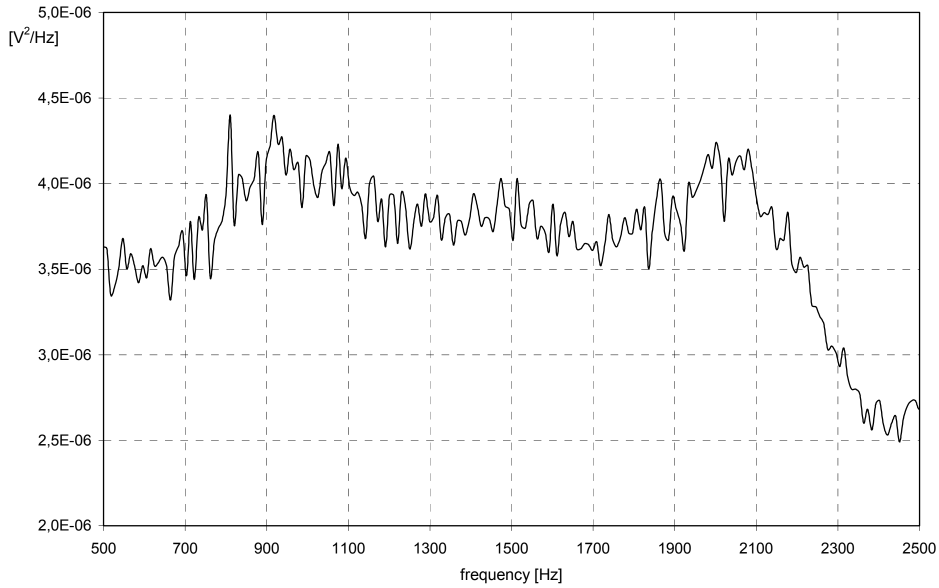
where $B_e = \frac{Q}{N\Delta t}$ determines the resolution of the spectrum analysis.

The determined above uncertainty of measuring power spectrum density depends on the uncertainty of the registered data used in the estimation (connected with the errors of the hardware of the measuring system), including parameters of a/d conversion, and also on the method of estimating power spectrum density (Figure 1). The assessment of the uncertainty of a/d processing was determined on the basis of the data concerning the parameters of the measurement card DAQ given by the producer. These data give the boundaries of the intervals, in which errors are contained, with the inaccessible information about the types of probability distributions of these errors. It was assumed that the errors of power gain, shift, non-linearity, quantisation, voltage temperature drifts, jitter and the error of settling time to full-scale step have rectangular distributions, whereas the noise error- has Gauss distribution. In the case of a 16-bit DAQ card within +/-10 V processing and sampling frequency 20 kSamples/s the uncertainty coming from the power gain error has the greatest share in combined standard uncertainty (Table 1).

Table 1. Standard uncertainties connected with particular sources of a/d conversion errors

Sources of errors	Producer's specification	Standard uncertainty u [µV] (for $x(i\Delta t)$ = 5 V)
offset	397,2 µV	u_{OFF} = 229,3
gain error	0,05 %	289 ppm (u_G = 1443,25)
temperature drift	±2 ppm/°C	within temperatures 15^0-35^0C $u_T = 0$
differential nonlinearity error	±0,5 LSB	u_{NL} = 88,1
quantisation	±0,5 LSB	u_q = 88,1
noise	1 LSBrms	u_n = 305,17
settling time to full-scale step	±1 LSB in 50µs	u_U = 176,2
jitter	±5 ps	u_τ = 1,8
$u_{x(i\Delta t)} = \sqrt{u_{OFF}^2 + u_G^2 + u_{NL}^2 + u_q^2 + u_n^2 + u_U^2 + u_\tau^2} = 1{,}508 \quad [mV]$		

a)



b)

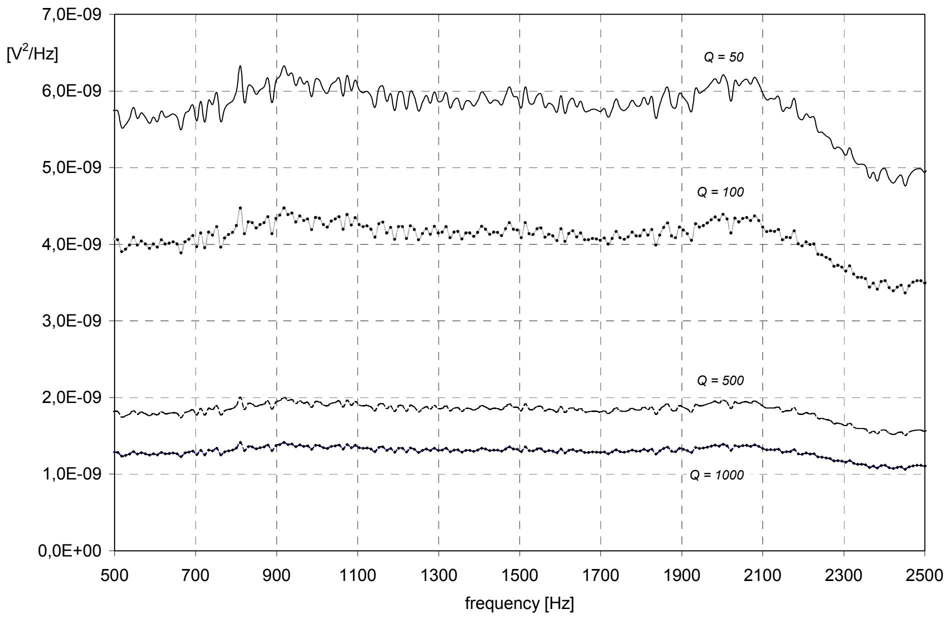


Figure 1. Exemplary waveform of the power spectrum density of the noise signal on the low frequency output of a communication receiver (a) and the corresponding to it combined uncertainty of the measurement (b), Q – the number of averagings

III. Uncertainty resulting from the estimator bias

The estimator of power spectrum density $\hat{G}_n(k)$ is unloaded when [4]:

$$E(\hat{G}_n(k)) = G_n(k) \tag{15}$$

The expected value of the power spectrum density estimator is determined by the dependence [5]:

$$E[\widetilde{G}_n(k)] = 2\Delta t \sum_{r=-(K-1)}^{K-1} \cos 2\pi f_r \Delta t (\frac{K-|r|}{K} R_{kr}) \quad (16)$$

where R_{kr} is autocorrelation function of signal.

The estimator bias $b(\hat{G}_n(k))$ is described by the equation [4]:

$$b(\hat{G}_n(k)) = E[\hat{G}_n(k)] - G_n(k) \quad (17)$$

This error depends on the properties of the examined random waveform. For an ergodic random waveform it can be made clearer by the following expression [4]:

$$b(\hat{G}_n(k)) \approx \frac{1}{24}(\frac{Q}{N\Delta t})^2 \hat{G}_n^{"}(k) = \frac{B_e^2}{24}\hat{G}_n^{"}(k) \quad (18)$$

where $\hat{G}_n^{"}(k)$ is the second derivative of the function $\hat{G}_n(k)$ in relation to a variable f_k.

For the white noise the estimator of power spectrum density is unloaded.
Assuming that this error has such distribution of the assumed values, which can be approximated by a fragment of the function of the density of normal distribution and considering its boundary values, uncertainty resulting from the estimator bias can be expressed as standard deviation of this distribution. When the changes in the waveform of the function in the considered range, are slight, it is possible to assume rectangular distribution. Then, uncertainty connected with the estimator bias $u_b(\hat{G}_n(k))$ is determined in the following way [2]:

$$u_b(\hat{G}_n(k)) = \frac{|\ b(\hat{G}_n(k))\ |}{\sqrt{12}} \quad (19)$$

The choice of the accepted assumptions concerning the character of this distribution depends on the experimental results of the power spectrum density of the examined random waveforms (Figure 2).

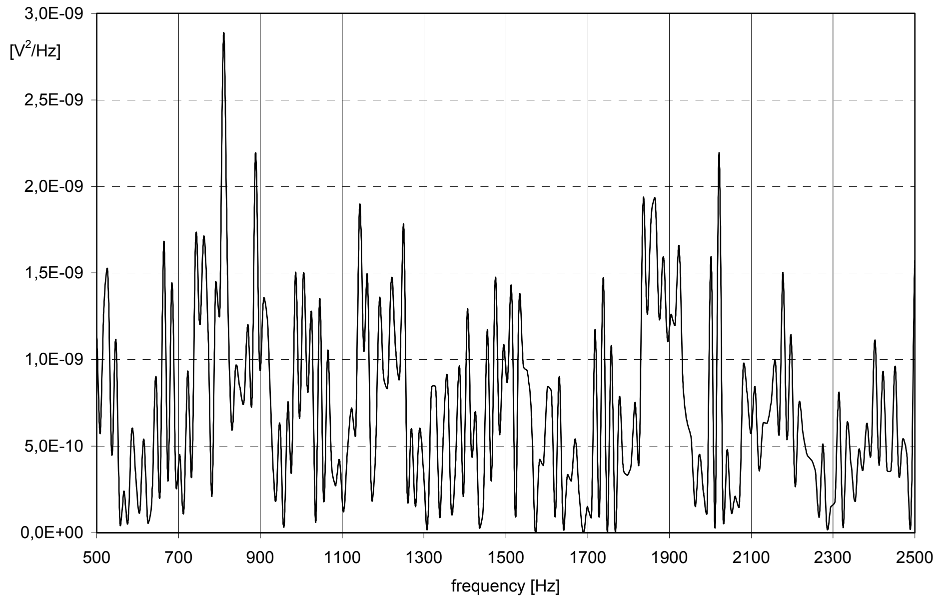


Figure 2. Uncertainty connected with the estimator bias of the power spectrum density of the noise signal on the low frequency output of a communication receiver

For the determined length of the observed signal ($N\Delta t$) a great number of averagings Q (more effective smoothing of the estimator) causes an increase in this component of uncertainty, which is the result of the worsened resolution of the spectrum analysis. Uncertainty connected with the estimator bias depends not only on the analysis parameters but on the properties of the analysed signal as well, especially when sharp local extremes appear in the power spectrum density waveform. (Figure 3).

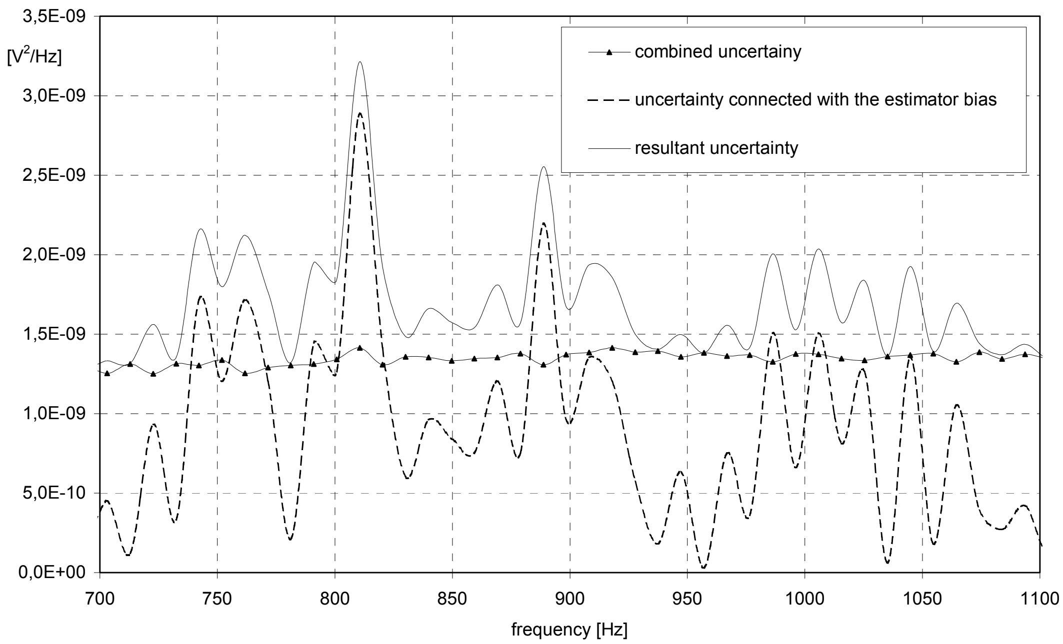


Figure 3. Component uncertainties of measuring the power spectrum density of the noise signal on the low frequency output of a communication receiver

IV. Summary

The use of the operation of smoothing the estimator of power spectrum density indeed decreases combined uncertainty in measuring power spectrum density, but at the same time there appears an increase in partial uncertainty resulting from the estimator bias. For a given uncertainty associated with signal samples recording, uncertainty in measuring power spectrum density depends on the accepted estimator and the appropriate selection of the parameters of the signal spectrum analysis (resolution, the number of samples). The presented dependences constitute the basis for calculations and simulations examining the influence of particular partial uncertainties associated with the parameters of the measurement system hardware and the accepted method of estimating the measured value (the applied DSP algorithm) on the value of uncertainty in measuring power spectrum density of a random waveform.

References

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