

Some Remarks On Measurement Of Time

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Abstract—In the paper some interpretation subtleties of a clock indication and clocks' comparison are discussed. The same amount of accumulated phase is common to a “second” of different clocks. As we can measure short and long time intervals with the same (or very similar) absolute uncertainty and use a reproducible phenomenon having very short duration in comparison to the length of the interval measured therefore properly designing the measurement we can get very small uncertainty of interval's measurement. It is explained why making the repetition series of the phenomenon longer improves the quality of averaging. Substitution of the difference of times of two clocks in the same instant and the time interval between two instants marked by each clock with the same date is justified.

I. Introduction

It is commonly recognized that the answer to the question “What is time?” (if the answer is possible at all) is rather expected from philosophy not from science. But it is science's responsibility to investigate the time flow, to measure the amount of time. In some respect the situation is similar when you consider the subtle idea of “information” in general and much better defined concept of “amount of information” in particular. Isaac Newton's attitude to time was rather transcendental: “Absolute, true and mathematical time, of itself, and from its own nature, flows equably, without relation to anything external” [1]. Albert Einstein simplified the idea of time linking time directly with measurement. According to Einstein time may be identified with an indication of a clock [2]. As a consequence every phenomenon affecting clock's indication influences the course of time.

Therefore in the paper marking time by a clock is considered. Then some aspects of uncertainty involved in time interval measurement are discussed. Some attention is paid to the subtleties of the measurement of time difference.

II. What does a clock indicate?

A clock is an instrument showing a count of number of a continuously repeating (reproducible) phenomenon, which is usually generated within the clock's machinery. It is postulated that the duration of every phenomenon in a series is the same. One can assume that the completion of every phenomenon (or some number of phenomena) augments the phase of a clock by 2π , which results in the clock's tick. The time $T(t)$ indicated by a clock (time function generated by a clock) and the clock's total instantaneous phase $\Phi(t)$ are related by [3]:

$$T(t) = \frac{\Phi(t)}{\omega_{nom}} \quad (1)$$

where t is an ideal time (practically some reference time), and ω_{nom} is the nominal angular frequency of the clock.

Let us interpret the relation (1). The nominal angular frequency of the clock is a conversion factor between the clock's time and the clock's instantaneous phase. If the clock ran with the nominal angular frequency, an increase of its phase during a unit of ideal time would be equal in numbers to ω_{nom} . But usually a given clock does not run with nominal frequency, thus the clock's phase increments in amount by ω_{nom} during a unit interval of that clock's time. Nevertheless, we give the same name for a given clock's time unit interval as we use for corresponding unit interval (that during which the same amount of phase is accumulated) of ideal time (reference time). So, what is common to a “second” of a given clock and the second of an ideal clock (or a second of a reference time, e.g. SI second) is the amount of accumulated phase but not the duration.

Thus a clock indicates instants (producing clock's ticks) when some amount of the clock's phase is accumulated and we designate these instants using the names of corresponding instants – featuring with the same amount of accumulated phase – of ideal clock (or rather some reference time scale). So a clock

shows the time $T(t)$ that would be the ideal time t if the clock's phase increase was performed with the nominal rate ω_{nom} . Because, however, the real rate of the clock's phase increase usually differs from the nominal rate, and varies in time, therefore the indications of the clock under considerations will not be normally in accordance with ideal time: mostly $T(t) \neq t$.

III. Uncertainty of time interval measurement

A small uncertainty of time interval measurements may be obtained when the time interval measured is long in relation to the duration of reproducible phenomenon. Let us use, for example, the sinusoidal waveform to measure a time interval equal to NT_0 , where T_0 is the duration of one cycle of the waveform (single phenomenon). To fulfill that measurement, we first detect with uncertainty

$$\Delta T_0 = \pm \frac{T_0 a}{2\pi A} \quad (2)$$

the instant of zero crossing starting the measurement. It is assumed that $\pm a$ is the uncertainty of identifying the value 0 V taken by the sinusoidal waveform and A is the waveform's amplitude. Then we count N cycles (periods), what in principle we can do without error.

Finally, we detect the instant of zero crossing completing the measurement, with the same uncertainty as previously. Relative uncertainty of measurement of time interval NT_0 is equal to

$$\frac{\Delta(NT_0)}{NT_0} = \pm \frac{1}{N} \frac{a}{\pi A} \quad (3)$$

where $\Delta(NT_0)$ is the measurement uncertainty of time interval NT_0 .

The formula (3) may be also interpreted as the average relative error of measurement of period T_0 ; the average is computed for N periods.

Let us notice, that N -times longer time interval is measured with N -times less relative uncertainty. It is so because the measurements of time interval T_0 and time interval NT_0 are carried out with the same absolute uncertainty, and determining the fact of generating N cycles (reproducing the phenomenon N times) requires only a qualitative ability to recognize the completing of the cycle (the occurrence of the phenomenon).

Let us consider an example. If $a/A = \pm 10^{-3}$, then according to (3) the relative uncertainty of measurement of single period's duration ($N=1$) is equal to $\pm 3.2 \cdot 10^{-4}$, but the relative uncertainty of measurement of duration of 10000 periods ($N=10^4$) equals $\pm 3.2 \cdot 10^{-8}$.

The uncertainty of measurement of a given time interval varies inversely proportional to the frequency of repetitions of a phenomenon if the relative uncertainty of determining the duration of a single phenomenon does not change with obviously shorter duration of a single phenomenon (we assume that there are no breaks in a course of repetitions of the phenomenon – repetitions without dead time). It is also a conclusions resulting from (3) when N increases because the duration T_0 of a single phenomenon is being shorter.

Even if we assume that the absolute uncertainty of measurement of a single period is equal to its duration (which actually means that a measurement system in a given moment shows only the number of completed cycles – reproduced phenomena – with accuracy ± 1), then a high frequency of reproduced phenomena allows us to obtain a small uncertainty of time interval measurement. The very definition of second in the SI system implies its relative uncertainty less than $1.09 \cdot 10^{-10}$. At present, SI second may be produced with uncertainty of the order of $1 \cdot 10^{-15}$ (information from the year 2000) [5].

So far we have assumed that we have at our disposal a standard generating a sequence of ideal reproductions of a phenomenon with duration of a single phenomenon equal to T_0 , and the uncertainty of time interval measurement results from our limited ability to recognize the beginning and the end of the phenomenon.

Actually, the duration of successively reproduced phenomenon is not unchanged. The influence of this different duration on time measurement may be minimized by averaging in time. This averaging should be understood in the following way [4]. Let T_k denote the duration of k -th reproduction of a phenomenon.

Then N reproductions determine time interval $\sum_{k=1}^N T_k$. The interval determined has such property that the absolute value of relative difference of this time interval and time interval equal to NT_0 , at sufficiently large N , satisfies relation

$$\frac{\left| \sum_{k=1}^N T_k - NT_0 \right|}{NT_0} < \frac{|T_k - T_0|}{T_0} \quad (4)$$

for almost every k . Formula (4) expresses the conviction that a longer time interval should be measured with a less relative uncertainty than a shorter interval. From (4) we obtain further

$$\left| \left(\frac{1}{N} \sum_{k=1}^N T_k \right) - T_0 \right| < |T_k - T_0| \quad (5)$$

for almost every phenomenon. Thus, the average value of duration of particular reproductions of a phenomenon is closer to the duration of ideal reproduction than the duration of most reproductions. Due to the fact that the durations of successive reproductions of a phenomenon are not independent from one another, the difference between their average value and the duration of ideal reproduction does not approach zero with increasing N . There is an optimal value for N (optimal time of averaging), increase of which does not make the effect of averaging better – does not bring closer the average value of duration of particular reproductions to the duration of ideal reproduction (ideal phenomenon) [5].

IV. Difference of times

The measurement of difference of clocks' times is involved in subtle interpretations, which are not always discerned. They will be demonstrated in the following example.

Let us consider two clocks located in the same place (e.g. in a laboratory), clock 1 indicating time $T_1(t)$ and clock 2 indicating time $T_2(t)$. We also assume access to an ideal clock showing time t (reference time).

To simplify description let us assume that in some starting instant clocks' times were set to zero: $T_1 = T_2 = t = 0$. For example let us consider now an instant, in which ideal clock indicates time $t=25$ s, clock 1 indicates time $T_1 = 27$ s₁, and clock 2 indicates time $T_2 = 24$ s₂. Let us notice that we have here three different seconds: "true" second s if ideal clock, second s_1 of clock 1, and second s_2 of clock 2. Second s_1 lasts shorter than the "true" second, second s_2 lasts longer. In the instant considered the difference between indications of clock 1 and clock 2 is equal to 3 second. A question arises: what seconds? We have three different ones. Are they seconds s , s_1 or s_2 ? In fact the difference between two clocks is in number of clock's ticks which are considered as second ticks. We have previously argued that the phase increment between second ticks is the same for each clock. So we can say that "phase second" is common second of clocks. In our example we may interpret the difference between clocks' indications as the difference of 3 "phase seconds". The person reading out the both clocks is most often ready to accept that the difference between their indications is equal to 3 s, as it is generally assumed that a clock measures "true" seconds.

When the difference of clocks' time is measured we are interested in the time difference value $T_1(t) - T_2(t)$ in a defined instant of t . However not in every instant, due to distortions and/or form of a clock's signal (timing signal), the clock's time is well determined. The instants of the best identification of clock's time are the instants of zero crossings of timing signal. Therefore the measurement of clocks' times difference is usually carried out by measuring the time passing away between corresponding zero crossings of the clock's 1 and the clock's 2 timing signals. Then the result of the measurement is associated with the instant t_p beginning the measurement, or with the instant t_k ending the measurement. In Fig. 1 we see that the value of the time difference of two clocks in a given instant and the value of the time passing away between two instants marked by each clock with the same date (e.g. 24 s) can be well matched if the instants considered are close enough. We can infer following relations based on Fig. 1:

$$T_1(t_k) - T_2(t_k) = (1+a)(t_k - t_p) \quad (6)$$

and

$$T_1(t_p) - T_2(t_p) = (1-b)(t_k - t_p) \quad (7)$$

where a and b are nonnegative and usually $a, b \ll 1$.

From (6) and (7) it follows that in the considered model of clocks (uniform, although not the same, flow of time of both clocks) the length of the time interval $t_k - t_p$ informs us on the difference of clock's times in the instant t_k , $T_1(t_k) - T_2(t_k)$, with inaccuracy a and on the clock's time difference in the instant t_p , $T_1(t_p) - T_2(t_p)$, with inaccuracy b . In a case of high quality clocks, a and b are very small and then the right-angled triangles $T_1(t_p)T_2(t_k)T_1(t_k)$ and $T_2(t_p)T_1(t_p)T_2(t_k)$ are very close to isosceles ones as well as the quadrangle $T_1(t_p)T_1(t_k)T_2(t_k)T_2(t_p)$ is very close to the parallelogram. So the measured time interval $t_k - t_p$ is almost equal to the time's difference $T_1(t) - T_2(t)$ of two clocks in a given instant t for $t \in [t_p, t_k)$ and even instants in immediate vicinity of this interval.

Since

$$T_1(t_p) = T_2(t_k) \quad (8)$$

we have

$$T_1(t_k) - T_2(t_k) = T_1(t_k) - T_1(t_p) \quad (9)$$

and

$$T_1(t_p) - T_2(t_p) = T_2(t_k) - T_1(t_k) \quad (10)$$

We can interpret the difference $T_1(t_k) - T_1(t_p)$ as a result of measurement of time interval $t_k - t_p$ by means of clock 1 (the faster by the assumption). This measurement according to (9), gives us an accurate information on the difference of clocks' times in the instant t_k , ending the measurement.

Similarly, the difference $T_2(t_k) - T_2(t_p)$ is a measurement result of the same interval $t_k - t_p$, but using clock 2 (the slower one). This result, according to (10), is equal to the clocks' time difference in the instant t_p , starting the measurement.

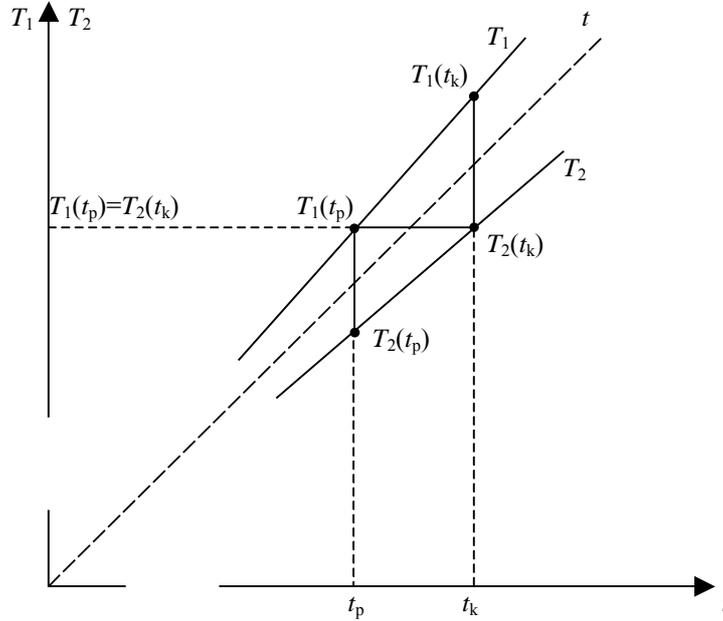


Fig. 1. The relation between the difference of times of two clocks in a given instant, $T_1(t) - T_2(t)$, and the time interval $t_k - t_p$ between two instants marked by two clocks with the same date, $T_1(t_p) = T_2(t_p)$

V. Conclusions

All clocks during time interval units having the same name (e.g. second) accumulate the same amount of phase. So “phase second” is the common second of all clocks. The duration of this unit is specific to each clock. It is possible to achieve very small relative uncertainty of measurement of a time interval as we can measure short and long intervals with the same (or very similar) absolute uncertainty and we are able to find a reproducible phenomenon having very short duration in comparison to the interval length measured. Making the repetition series of the phenomenon longer improves the quality of averaging. There is closeness of the agreement between the difference of times of two clocks in the same instant and the time interval between two instants marked by each clock with the same date. This closeness is preserved for all instants within time interval considered and in its immediate vicinity. Using one of two compared clocks for measuring the considered time interval we get exact information on the clock's time difference in the instant starting or ending the measurements (it depends on a clock chosen).

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