Some Results on the Validation of Widrow's Model for Sinusoidal Signals

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Abstract – Some results on the validity of Widrow's model of quantizing sinusoidal signals in a uniform quantizer are presented. Sripad and Snyder's formulation is used to derive a general equation for the probability density function of the quantization noise, for a generic sinusoidal signal, which is applied to different types of sinusoids. The equations obtained in this way are numerically evaluated to characterize the quantization noise by means of its average and variance. In this calculation, a novel method to accelerate the series convergence has been used. The results obtained provide a confirmation that Widrow's model can only be applied to sinusoids when their amplitudes are much larger then the quantization step.

I. Introduction

Widrow's model [1], also known as uniform quantization-error model, is generally considered to be the most suitable to represent the effect of quantization by a uniform, nonsaturated quantizer and has become a standard in the representation of analog-to-digital (ADC) noise. It represents the effect of the quantizer in terms of an additive quantization noise (QN) *E(t)*, in the form:

$$X_q(t) = X(t) + \mathrm{E}(t) \tag{1}$$

where *X(t)* is the quantizer input signal and $X_q(t)$ the quantizer output signal. *E(t)* is a random process uniformly distributed between $\pm\Delta/2$, Δ being the quantization step; it has mean value 0 and variance $\Delta^2/12$, it is uncorrelated with the input signal *X(t)*, and has a white spectrum. When the model is valid, nonlinear systems behave like linear ones.

In order for the model to be valid, it is necessary that the signals being quantized obey several quantization conditions (QC) [1], all related to their probability density function (PDF) and characteristic function (CF). Those QC are so restricted that no real signal can fulfil them completely. The validity of the model must then be ascertained on a case-to-case basis. In this context, Sripad and Snyder [2] have specified a necessary and sufficient condition for the quantization noise to be uniform and white and have introduced an equation for the PDF of the real noise, for a given signal, thus enabling us to validate the model. Sripad and Snyder applied their theory to signals obeying Gaussian distribution.

As sinusoidal signals are the most used for testing of ADC's, the assessment of QN for this kind of signals is of crucial importance. Unfortunately, sinusoidal signals do not verify any of the quantization conditions and the QN must be determined. Thus, further applications of Sripard and Snyder's formulation are described in the literature: Wagdy e Ng [3] have applied this formulation to sinusoidal signals, with and without dither, to calculate the PDF and variance of the noise. Hejn and Pacut [4] have also applied the same formulation to a sinusoidal signal with an offset. In this paper we have used a more general approach, since we have started with the general definition of a sinusoidal signal, and then particularized to specific cases, in order to calculate the relevant noise parameters.

II. Theoretical background

The PDF of the quantization noise [2], *E(t),* is given by:

$$f(\varepsilon) = \begin{cases} \dfrac{1}{\Delta} + \dfrac{1}{\Delta}\displaystyle\sum_{n\neq 0} \Phi_x(2\pi\frac{n}{\Delta})\exp(-j2\pi n\frac{\varepsilon}{\Delta}), & \text{if } \varepsilon \in \left[-\Delta/2, \Delta/2\right[\\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where Φ_x is the characteristic function of the input *X*. From (2) we can deduce expressions for the mean error

$$E\{\varepsilon\} = \int_{-\Delta/2}^{\Delta/2} \varepsilon f(\varepsilon)d\varepsilon \tag{3}$$

and for the mean squared error

$$E\{\varepsilon^2\} = \int_{-\Delta/2}^{\Delta/2} \varepsilon^2 f(\varepsilon)d\varepsilon \tag{4}$$

and then for the variance of the error $\sigma^2_\varepsilon = E\{\varepsilon\}^2 - E\{\varepsilon^2\}$.

Consider the input signal of the uniform quantizer to be a random signal

$$X(t) = D(t) + A\sin(\omega t + \Theta)\,. \tag{5}$$

This stochastic process is the sum of two independent terms: The first, *D(t)*, is a generalized offset. The second term is the sinusoid $A\sin(\omega t+\Theta)$, where *A* represents the amplitude, $\omega=2\pi f$ is the angular frequency and Θ is the phase, assumed to be a random variable with uniform distribution in the interval $\left]-\pi,\pi\right[$. The PDF of the sinusoid is [5]

$$f(x)=\begin{cases}\dfrac{1}{\pi\sqrt{A^2-x^2}}, & \text{if } |x|<A \\ 0, & \text{otherwise}\end{cases} \tag{6}$$

Using a mathematical handbook [6], it can be shown that the characteristic function of the sinusoid is

$$\Phi(u)=J_0(Au)\,, \tag{7}$$

were $J_0(Au)$ is the Bessel function of the zero order and first kind. It is known that if two random variables are independent, then the density of their sum equals the convolution of their densities. From this result and from the convolution theorem we can state that if X an Y are two independent random variables, then

$$Z=X+Y \Rightarrow \Phi_z(u)=\Phi_x(u)\Phi_y(u)\,. \tag{8}$$

From (7) and (8) it follows that the characteristic function of the input signal is

$$\Phi_x(u)=J_0(Au)\Phi_d(u)\,. \tag{9}$$

were Φ_d is the characteristic function of $D(t)$. Substitution of (9) in (2) yields the PDF of the quantization noise of (5)

$$f(\varepsilon)=\begin{cases}\dfrac{1}{\Delta}+\dfrac{1}{\Delta}\sum_{n\neq 0} J_0(2\pi n\dfrac{A}{\Delta})\Phi_d(2\pi n\dfrac{A}{\Delta})\exp(-j2\pi n\dfrac{\varepsilon}{\Delta}), & \text{if } \varepsilon\in[-\Delta/2,\Delta/2[\\ 0, & \text{otherwise}\end{cases} \tag{10}$$

The above equation will take different forms, depending on the particular shape of the characteristic function of the generalised offset $D(t)$. We shall consider two: a) $D(t)=0$ in which case $X(t)$ becomes a pure sinusoid without offset; b) $D(t)=k$, were k is a positive constant in which case $X(t)$ becomes a sinusoid with a deterministic offset.

III. Sinusoid without offset

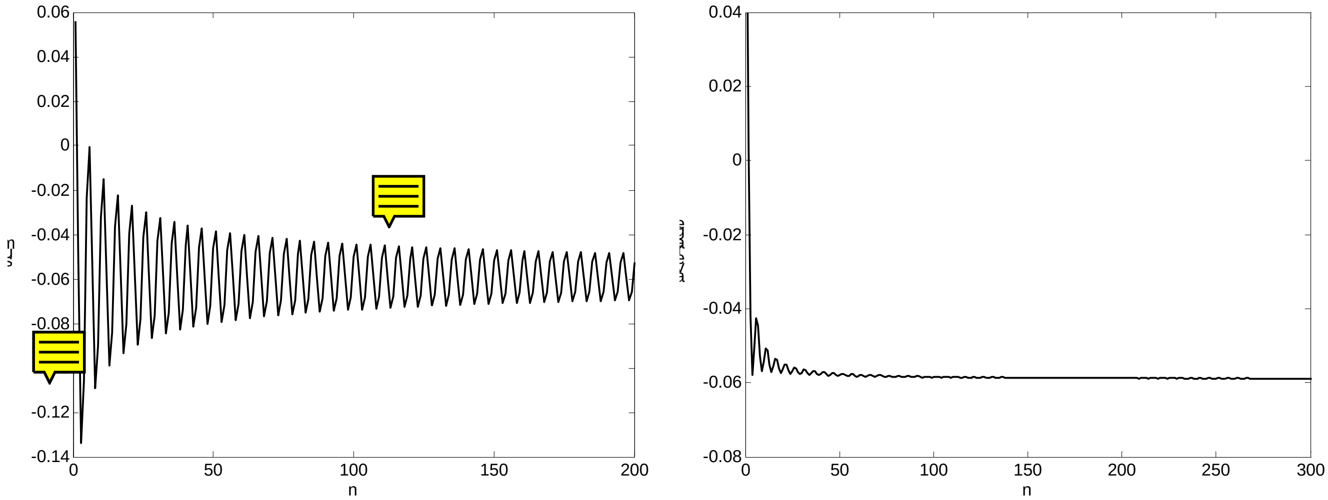
If $D(t)=0$, the PDF is $f(d)=\delta(d)$, and its characteristic function is equal to one ($\Phi_d(u)=1$). In this case the density function of the quantizer error is:

$$f(\varepsilon)=\frac{1}{\Delta}+\frac{2}{\Delta}\sum_{n=1}^{\infty} J_0(2\pi n\frac{A}{\Delta})\cos(2\pi n\frac{\varepsilon}{\Delta}),\quad \varepsilon\in[-\Delta/2,\Delta/2[\,, \tag{11}$$

it's mean is zero ($E\{\varepsilon\}=0$) and the variance,

$$\sigma_\varepsilon^2=E\{\varepsilon^2\}=\frac{\Delta^2}{12}+\frac{\Delta^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}J_0(2\pi n\frac{A}{\Delta}),\quad \varepsilon\in[-\Delta/2,\Delta/2[\tag{12}$$

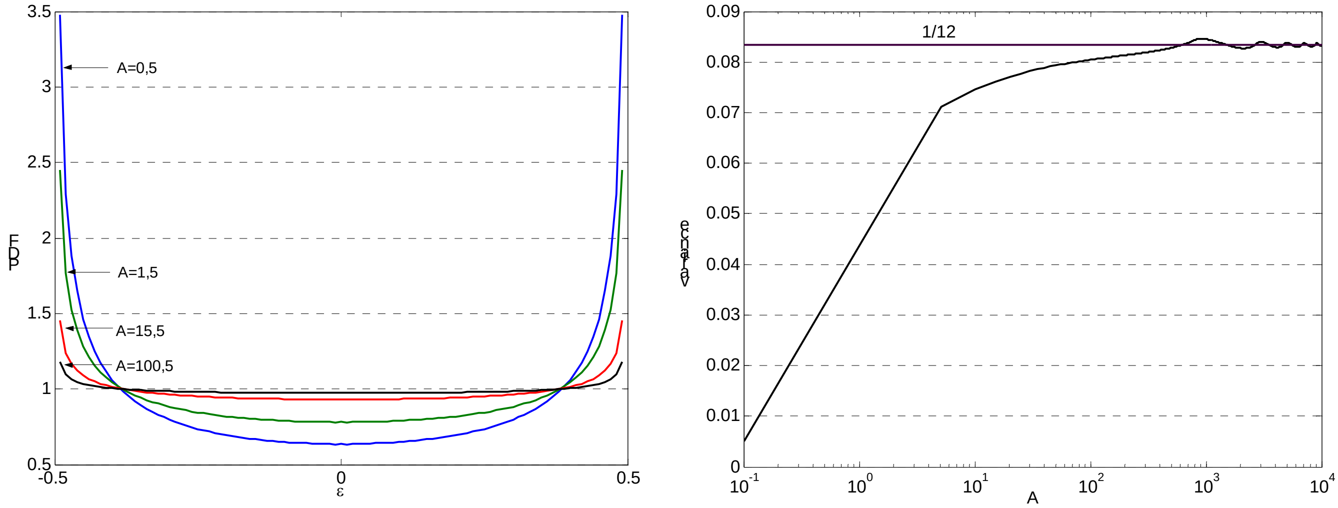
In order to evaluate how closely the uniform quantization-error model fits the actual density and to calculate the mean and variance, the series in (11) and (12) must be summed. It can be shown that these series are convergent [7], but the limits are unknown and must be evaluated numerically. In Fig. 1a) we plot the value of the series in (11) as a function of the number of terms n, for A=1.5 and ε=3, both normalized to the value of Δ, which is equivalent to set Δ=1. For different values of A and ε the behavior obtained is essentially the same. The series converges at a rate which depends in a complicated manner on the function parameters. In some cases, convergence has not been reached even after summing as many as 10000 terms. However, as Fig. 1b) also shows, the average over the partial sums converges much faster and has thus been used to evaluate the limit of the series in (11) and (12).



a) b)

Figure 1: a) The sum in (11) as a function of n, for A=1.5 and ε=3; b) Average of (11) versus n.

The relevant quantities were numerically evaluated using programs in Matlab and using C to sum the series to increase the speed of the computation. All parameters were normalized to Δ. Fig. 2 shows the results for a sinusoid signal without offset. The calculated mean error of the quantization noise is zero, but the PDF plot indicates that only for values of A/Δ grater than 100, can the noise distribution be considered uniform. The variance in Fig. 2b) approaches the theoretical value of $\Delta^2/12$ very slowly, being in the expected range only for A>1000, which is equivalent to consider a full-scale signal quantized in a 10 bits ADC. These results are similar to these obtained by Wagdy and Ng [3], one difference being that in this case the variance approaches the theoretical value in an oscillatory manner, while in Wagdy and Ng it decreases to 1/12 monotonically.



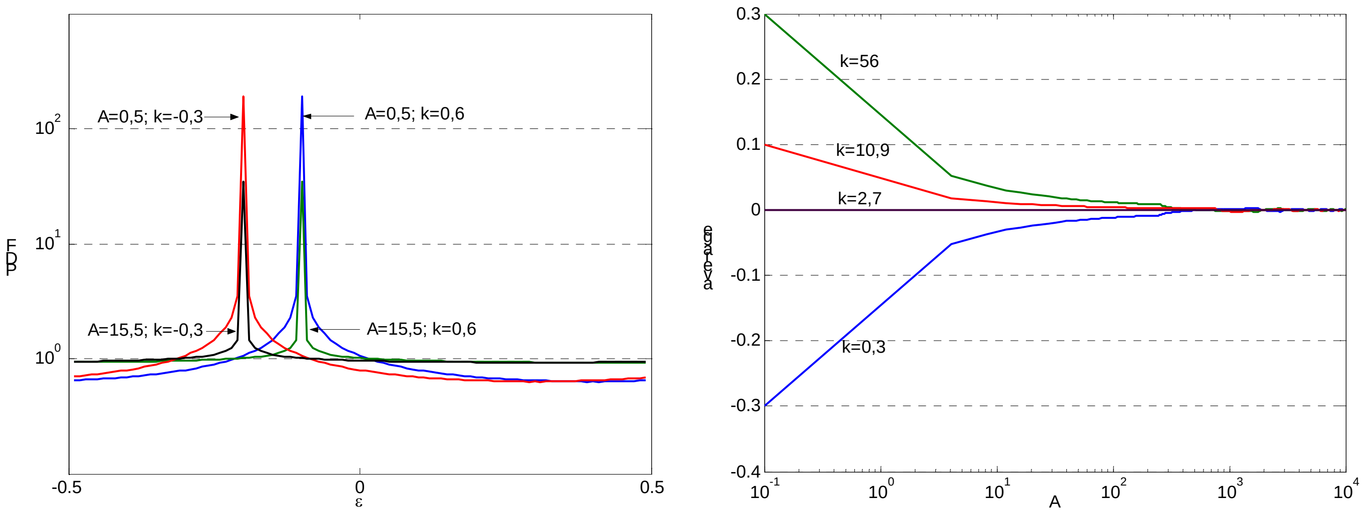
a) b)

Figure 2: a) PDF of the quantization noise for a sinusoidal signal without offset, for different values of *A* normalized to Δ; b) Variance of the quantization noise of the same signal, as a function of the normalized amplitude *A*.

IV. Sinusoid with constant offset

A similar approach to the one presented previously can be followed for $D(t)=k$, being k a positive constant, which is equivalent to consider the input process as a sinusoid with a constant offset. The probability density function of $D(t)$ is $f(d)=\delta(d-k)$, and the characteristic function is $\Phi_d(u)=exp(-jku)$. Using (2) and after some algebric manipulation, the PDF becomes

$$f(\varepsilon)=\frac{1}{\Delta}+\frac{2}{\Delta}\sum_{n=1}^{\infty}J_0(2\pi n\frac{A}{\Delta})\cos(2\pi n\frac{k+\varepsilon}{\Delta}),\quad \varepsilon\in\left[-\Delta/2,\Delta/2\right[\tag{13}$$



a) b)

Figure 3: a) PDF of the quantization noise for a sinusoidal signal with constant offset, for different values of *A* and *k* normalized to Δ; b) Mean of the quantization noise of the same signal, as a function of the amplitude *A*.

The mean is

$$E\{\varepsilon\} = \frac{\Delta}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}J_0(2\pi n\frac{A}{\Delta})\sin(2\pi n\frac{k}{\Delta}),\quad \varepsilon\in[-\Delta/2,\Delta/2[\,, \tag{14}$$

and the mean squared error is

$$E\{\varepsilon^2\} = \frac{\Delta^2}{12}+\frac{\Delta^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}J_0(2\pi n\frac{A}{\Delta})\cos(2\pi n\frac{k}{\Delta}),\quad \varepsilon\in[-\Delta/2,\Delta/2[\,. \tag{15}$$

Variance is obtained by subtracting (15) from the square of (14).

Expressions (12-14) are evaluated using the method to sum the series explained above. In Fig. 3 we show the results for a sinusoid with constant offset for different values of k, A and ε. These results show that the PDF of the quantization noise can hardly be considered uniform. Nevertheless both the mean in Fig. 3b) and the variance in Fig. 4 converge to their theoretical values, being reasonably close to their range for A>1000.

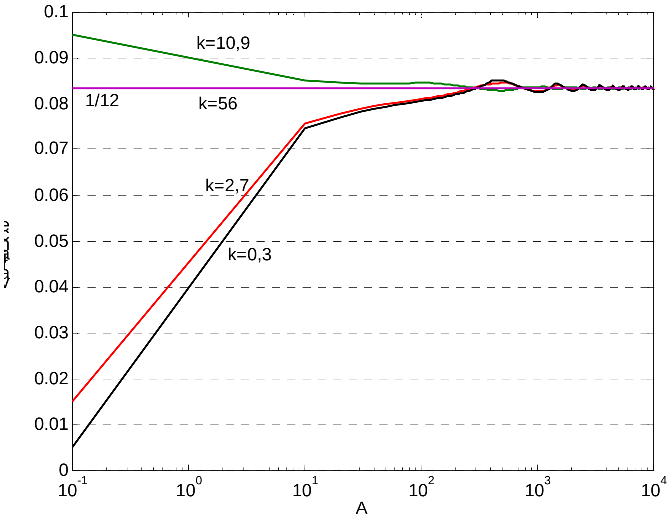


Figure 4: Variance of the quantization noise of a sinusoid whit constant offset, as a function of the normalized amplitude A, for different values of offset k.

If instead of A=0.5, 1.5, 15.5 and 100.5, in Fig. 2a) we use, for example, A=0.3, 1.3, 15.3 and 100.3, the plots are shifted and Fig. 2a) becomes similar to Fig. 3a). Furthermore, our results also show that characterizing the distribution by its average and variance can be misleading, i.e. distributions that may seem uniform from the average and variance point of view can in fact be show to be very far from uniform when one looks at the PDF.

V. Conclusions

A detailed study of the validity of Widrow's model for sinusoidal signals, in uniform nonsaturating quantizers, has been conducted following Sripad and Snyder's formulation, but using a generic sinusoid to determine a general equation for the PDF of the quantization noise. The mean and the variance of the noise have been calculated using a novel method to accelerate convergence in several particular cases. The results confirm that Widrow's model is only valid for sinusoidal signals with amplitudes several times larger then the quantization step.

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