Toward a Standardized Multi-Sinewave Fit Algorithm

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***Abstract*—** Multi-sinewave test methods require algorithms for multiple-tone parameter estimation. There exist a vast amount of publications on the topic [1]. This paper presents a generalization of the IEEE four-parameter sinewave fit algorithm suitable to handle data comprising multiple sinewaves. The proposed method directly estimates the $3p+1$ parameters of a p-tone model. The algorithm is analyzed numerically with emphasize on its convergence properties and statistical efficiency. The initialization of the algorithm is of major importance and an attempt to formulate a proper initialization procedure is presented.

I. Introduction

Sinewave test methods have for a long period of time been dominating in testing digital devices. The extraction of the parameters of a single tone is well known, and there exists a standardized method [2], [3]. There are several physical parameters that can not be measured by single-sinewave tests. One example is the inter-modulation test for analog to digital converters (ADCs) [2], [3]. In the inter-modulation test, one must not only estimate the parameters of the two tones that excite the ADC, but also the number of harmonics introduced by the nonlinearity of the ADC.

Unfortunately, it is more difficult from a measurement to resolve multiple sinewaves than a single one. If one ignores the fact that several sinewaves are present and use an estimator designed for a single tone the estimates get biased due to spectral leakage [4]. Several researchers in the area have tried to resolve this problem in several different ways. One attractive method is to use the maximum likelihood (ML) estimator [5]. However, there exists no closed form solution for the multi-tone model due to the highly nonlinear criterion function. There exist some iterative approaches to solve the ML problem, among others the ones in [6] and [7], respectively. The accuracy of these methods reaches the corresponding Cramér-Rao bound (CRB). The one recently proposed in [7], however, has superior SNR threshold than the method in [6].

The aim of this paper is to review some results on frequency estimation and present their implications on the instrumentation and measurement set-up. Further, a generalization of the four-parameter fit of [2], [3] is presented, that is able to estimate the $3p+1$ parameters of a multi-tone model.

II. Requirements on a sinewave fit algorithm

The multi-tone estimation problem may be separated into subproblems like signal detection, algorithm initialization and parameter extraction. In the instrumentation and measurement set-up, we are typically interested in detection of line spectral components above the noise floor (that is, the spurious frequencies), as well as the level of the noise floor itself.

A. Cramér-Rao bound and signal model

In the literature, a complex-valued signal model is often employed. At first glance, this complex-valued model seems to make the analysis more complicated than using a real-valued model. In fact, the opposite holds true that can be seen, for example, from the exact CRB for a single complex-valued exponential signal (or, cisoid) disturbed by additive white Gaussian noise. A well known result on the achievable accuracy of any unbiased estimator of the normalized angular frequency ω for evenly sampled data is given by [8]

$$\mathrm{Var}(\hat{\omega}) \geq \frac{6}{\mathrm{SNR}\, N^2\,(N-1)} \tag{1}$$

where $\hat{\omega}$ denotes an estimate of ω. Here and from now on, N denotes the number of available samples. Further, SNR denotes the signal-to-noise ratio. In the real-valued case, the corresponding CRB is more complicated and results in an expression that is dependent on the signal frequency and the initial phase [9]. However, an asymptotic (as $N \to \infty$) expression is known to be

$$\mathrm{Var}(\hat{\omega}) \geq \frac{12}{\mathrm{SNR}\, N^3}. \tag{2}$$

Moreover, in a test environment without I/Q-modulation there exclusively exist real-valued signals which make a complex-valued signal model improper. In this paper, the following multi-sinewave signal model is employed

$$s[n] = C + \sum_{\ell=1}^{p} A_\ell \cos \omega_\ell t_n + B_\ell \sin \omega_\ell t_n, \quad n = 1, \ldots, N \tag{3}$$

where t_n denotes the (normalized) sampling instants. The parameters A_ℓ, B_ℓ and C are all assumed to be unknown constants. The constant angular frequencies ω_ℓ are also considered as unknown parameters. Here, $\omega_\ell = 2\pi f_\ell / f_s$ where f_ℓ is the signal frequency in Hertz and f_s is the sampling frequency. In (3), regular sampling at f_s Hertz corresponds to an integer $t_n = n$. The number of sinewaves p is assumed to be known. This is a reasonable assumption in many applications where the number of sought waveforms is known. Estimation of p is further discussed in Section III-A. The model (3) is equivalent with modeling each tone as an amplitude- and phase shifted sinusoid, that is

$$s[n] = C + \sum_{\ell=1}^{p} \alpha_\ell \sin(\omega_\ell t_n + \phi_\ell) \tag{4}$$

where $A_\ell = \alpha_\ell \sin \phi_\ell$ and $B_\ell = \alpha_\ell \cos \phi_\ell$.

The measured signal $x[n]$ is a sum of the signal (3) and an additional noise term $w[n]$, that is

$$x[n] = s[n] + w[n], \quad n = 1, \ldots, N. \tag{5}$$

The noise is assumed to be zero-mean white Gaussian with variance σ^2. The assumption that $w[n]$ is Gaussian is restrictive, but is, on the other hand, only used in order to assess the performance of the algorithm by a comparison with the CRB. If the Gaussian noise hypothesis fails it is shown in [10] that a least-squares fit asymptotically results in an efficient estimator. Further, if the covariance matrix of the estimates only depends on the second order statistics of the data, a least-squares fit will result in the minimum-variance estimate [6].

B. Frequency resolution

Frequency resolution is an important topic that has to be discussed in some detail. In general terms, resolution of two line spectral components is a function of the separation between them, the amplitudes, as well as the level of the noise floor. We define frequency resolution as the minimum angular frequency separation between two neighboring sinewaves $\Delta\omega = |\omega_i - \omega_j|$ for which both spectral components can be detected and then estimated accurately. We emphasize that frequency resolution in model based estimation differs from the Fourier or Rayleigh resolution, roughly determined by [11]

$$\Delta\omega = \frac{2\pi}{N}. \tag{6}$$

Accordingly, proper high-resolution methods resolve line spectral components within the Fourier resolution. Accurate estimation of closely-spaced tones has been presented in [12]. At SNR $= 10$dB and for $N = 100$, the method in [12] is shown to resolve two equipowered sinewaves with frequencies as close as $\Delta\omega = \pi/(2N)$, that is the fourth of the Fourier resolution (6). Further, the method is shown to resolve two sinewaves whos amplitudes differ by 10dB at half the Fourier resolution.

C. Performance of multi-tone methods

The performance one can expect from a proper method is a relevant topic. The performance depends on several causes, but the two most important items are to detect the correct number of tones and the initialization procedure for the fine-tuning of the parameter estimator. If detection and initialization are performed in a correct and successful way the error variance of the overall method is expected to be close to the CRB. If the spectral components are well separated in frequency each parameter estimate is expected to reach its corresponding CRB, that for the frequency approximately coincides with the single tone CRB in (2). However, if two sinewaves are closely located in frequency the single tone assumption is not valid. In [13], it is shown that if the frequency separation $\Delta\omega$ is larger than about 1.5 times the Fourier resolution (6) then the CRB in the dual tone case basically coincides with (2).

III. A generalized IEEE 1057 algorithm

In [7], an algorithm for multi-tone parameter estimation based on the IEEE Standard 1057 four-parameter fit in combination with the expectation-maximization (EM) algorithm is presented. The

EM-algorithm was employed to separate the measurements into p single tone components that were the input to p parallel four-parameter fits. Here, an alternative to the method in [7] is derived, given by an extension of the four-parameter fit to a 3p+1-parameter fit, where p denotes the number of spectral components.

The proposed algorithm can be divided into two steps. First, the individual spectral components are detected and initial estimates are formed. The second step increases the accuracy of the estimates by successive iterations. Each step is crucial in order to obtain a fully automated multi-sinewave estimator. In this Section the attention is concentrated on the second iterative fine-tuning step.

A. A procedure for algorithm initialization

A general issue in non-linear parameter estimation is the threshold-effect that occurs at a certain SNR, below witch the estimates are deteriorated. In the considered case, this threshold depends on the number of data samples [14]. In [15], an indicator quantity $\gamma = N\,\mathrm{SNR}/\log_e N$ was introduced. It was shown (by aid of the Barankin bound) that $\gamma \geq 70$ always pull the single tone ML estimator out of the threshold region. The initialization can be performed by searching for the p largest peaks in the periodogram. There exist fast and efficient methods to compute the periodogram and therefore this is an attractive approach. A drawback with a periodogram-based method is the poor frequency resolution, determined by the Rayleigh resolution. Another drawback with the periodogram is that a strong sinewave shadows weaker ones. This masking effect results in an inferior frequency resolution than (6). When the periodogram is successful in resolving the individual sinewaves one may expect an estimation accuracy resulting in a mean-squared-error (MSE) of (for $\ell = 1, \ldots, p$)

$$\mathrm{MSE}(\hat{\omega}_\ell) = \mathrm{E}\{(\hat{\omega}_\ell - \omega_\ell)^2\} = \frac{\pi^2}{3N^2}. \tag{7}$$

The MSE in (7) results in a root-MSE (RMSE) of order $2/N$. The RMSE indicates the size of the minimum convergence radius of any iterative algorithm used for fine-tuning of the estimates.

Given initial values $\{\hat{\omega}_\ell\}$, estimates of the unknown $\{A_\ell\}$, $\{B_\ell\}$ and C can be found by solving a linear system of equations. Using a vector notation, the signal $s[n]$ in (3) can be written as

$$s = \mathbf{H}\theta, \tag{8}$$

where s is the signal vector

$$s = \begin{bmatrix} s[1] & \ldots & s[N] \end{bmatrix}^T, \tag{9}$$

(T denotes transpose) and θ the parameter vector

$$\theta = \begin{bmatrix} C & A_1 & B_1 & A_2 & B_2 & \ldots & A_p & B_p \end{bmatrix}^T. \tag{10}$$

Further, $\mathbf{H}$ is a $N \times 2p+1$ matrix given by

$$\mathbf{H} = \begin{bmatrix} 1 & \cos\omega_1 t_1 & \sin\omega_1 t_1 & \ldots & \cos\omega_p t_1 & \sin\omega_p t_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \cos\omega_1 t_N & \sin\omega_1 t_N & \ldots & \cos\omega_p t_N & \sin\omega_p t_N \end{bmatrix}. \tag{11}$$

Given initial estimates $\{\hat{\omega}_\ell\}$ for $\ell = 1, \ldots, p$ the in least-squares sense optimal estimate of θ (10) is given by (if the matrix $\mathbf{H}^T\mathbf{H}$ is invertible)

$$\hat{\theta} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T x. \tag{12}$$

In (12), $\mathbf{H}$ is formed by plugging in the $\{\hat{\omega}_\ell\}$ into (11), and the vector x contains the measurements, that is

$$x = \begin{bmatrix} x[1] & \ldots & x[N] \end{bmatrix}^T. \tag{13}$$

B. A $3p+1$ parameter fit algorithm

The algorithm is a generalization of the four-parameter sinewave fit algorithm [2], [3] to handle p-tone data. In the four-parameter sinewave fit the nonlinear signal is linearized around the previous frequency estimate resulting in a linear signal model. Given the frequency estimates $\hat{\omega}_\ell^{(r)}$ at the iteration step r, a Taylor series expansion around $\hat{\omega}_\ell^{(r)}$ can be performed as

$$\cos\omega_\ell t_n \approx \cos\hat{\omega}_\ell^{(r)} t_n - t_n \sin\hat{\omega}_\ell^{(r)} t_n \cdot \Delta\omega_\ell^{(r)} \tag{14}$$

and

$$\sin\omega_\ell t_n \approx \sin\hat{\omega}_\ell^{(r)} t_n + t_n \cos\hat{\omega}_\ell^{(r)} t_n \cdot \Delta\omega_\ell^{(r)}, \tag{15}$$

where $\Delta\omega_\ell^{(r)} = \omega_\ell - \hat{\omega}_\ell^{(r)}$. Inserting (14) and (15) in (3) results in the approximate signal model

$$s[n;\vartheta_r] \approx C^{(r)} + \sum_{\ell=1}^{p} A^{(r)} \cos\hat{\omega}_\ell^{(r)} t_n + B^{(r)} \sin\hat{\omega}_\ell^{(r)} t_n + \\ - \hat{A}^{(r-1)} \Delta\omega_\ell^{(r)} t_n \sin\hat{\omega}_\ell^{(r)} + \hat{B}^{(r-1)} \Delta\omega_\ell^{(r)} t_n \cos\hat{\omega}_\ell^{(r)}, \tag{16}$$

where the two estimates $\hat{A}^{(r-1)}$ and $\hat{B}^{(r-1)}$ from iteration $(r-1)$ have been inserted in the last two terms. In (16), the multi-sinewave parameter vector ϑ_r is given by

$$\vartheta_r = \begin{bmatrix} C^{(r)} & A_1^{(r)} & B_1^{(r)} & \Delta\omega_1^{(r)} & \ldots & A_p^{(r)} & B_p^{(r)} & \Delta\omega_p^{(r)} \end{bmatrix}^T. \tag{17}$$

Using the vector ϑ_r, (16) can be written using matrix notation as

$$s[\vartheta_r] \approx \mathbf{D}_r \vartheta_r. \tag{18}$$

The matrix $\mathbf{D}_r$ forming the set of basis functions is given by

$$\mathbf{D}_r = \begin{bmatrix} \mathbf{1} & D_1^{(r)} & \ldots & D_p^{(r)} \end{bmatrix} \tag{19}$$

where the vector $\mathbf{1} = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T$ is of length N, and the sub-matrices $D_\ell^{(r)}$ are given by

$$D_\ell^{(r)} = \begin{bmatrix} \cos\omega_\ell^{(r)} t_1 & \sin\omega_\ell^{(r)} t_1 & -A_\ell^{(r-1)} t_1 \cos\omega_\ell^{(r)} t_1 + B_\ell^{(r-1)} t_1 \sin\omega_\ell^{(r)} t_1 \\ \vdots & \vdots & \vdots \\ \cos\omega_\ell^{(r)} t_N & \sin\omega_\ell^{(r)} t_N & -A_\ell^{(r-1)} t_N \cos\omega_\ell^{(r)} t_N + B_\ell^{(r-1)} t_N \sin\omega_\ell^{(r)} t_N \end{bmatrix}. \tag{20}$$

The least-squares solution of ϑ_r can be computed according to

$$\vartheta_r = \left(\mathbf{D}_r^T \mathbf{D}_r\right)^{-1} \mathbf{D}_r^T x. \tag{21}$$

For each iteration of (21) the frequency estimates $\{\hat{\omega}_\ell^{(r)}\}$ are updated according to

$$\omega_\ell^{(r+1)} = \omega_\ell^{(r)} + \Delta\omega_\ell^{(r)}, \quad \ell = 1, \ldots, p. \tag{22}$$

The iterations are stopped when sufficient precision is reached for some $\{\varepsilon_\ell\}$ according to $|\Delta\omega_\ell^{(r)}| \leq \varepsilon_\ell$ for all $\ell = 1, \ldots, p$. One should note that a direct calculation of (21) may be numerically imprecise and from an implementation point of view it is recommended to use some matrix factorization algorithm [16].

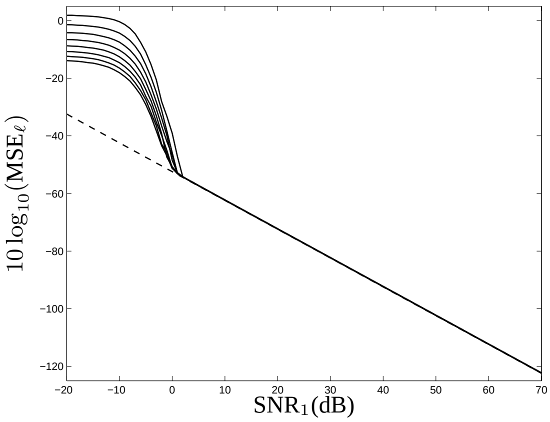
IV. Numerical evaluations

The proposed algorithm has been evaluated in four different scenarios, all based on equidistant sampling $t_n = 1, 2, \ldots$. The first two scenarios (*a* and *b*) have been evaluated using the initialization routine in Section III-A. In the latter two scenarios (*c* and *d*) the iterative algorithm is investigated (for simplicity) when initialization is performed around the true frequency, according to $\hat{\omega}_\ell = \omega_\ell + u$, where u is a random variable with uniform distribution in the interval $[-1/(2N), 1/(2N)]$. Here, the initialization is guaranteed to be within the expected resolution of a periodogram based initialization method. The number of data samples N is set to $N = 128$ in all scenarios. The normalized angular frequencies $\{\omega_\ell\}$ have been drawn from a uniform distribution within the interval $[0, \pi)$, which corresponds to a frequency between 0 and $f_s/2$ Hertz. The initial phases $\{\phi_\ell\}$ are chosen from a uniform distribution within the interval $[0, 2\pi)$. The amplitude settings in each scenario are given in Table 1.

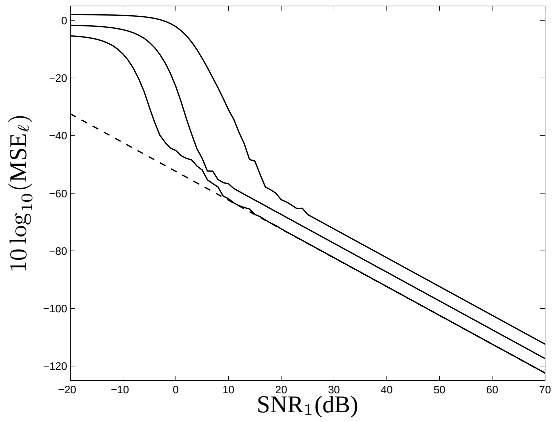
The results (based on 10^5 independent runs) are presented in Figure 1. For high SNRs, each frequency estimate has a corresponding variance that coincides with the CRB (2). For low SNRs, in cases *a*)–*b*) there is a distinct threshold where the algorithm fails to resolve the frequencies, at about SNR $= 0$dB. However, this threshold is inherent from the initialization where the periodogram method is used. In

Table 1
Parameter values used in the numerical evaluations.

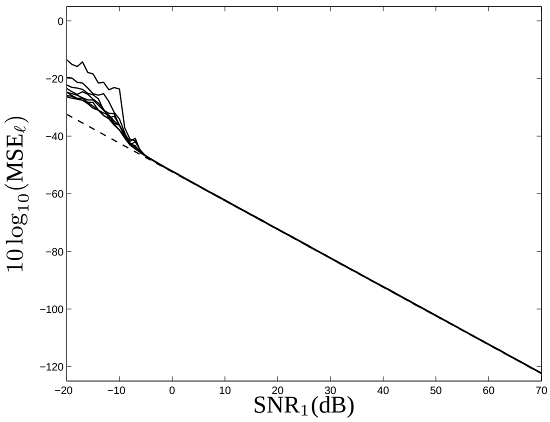
	p	α_ℓ
a)	8	$1 \ \forall \ \ell$
b)	3	$\{1, 10^{-1/4}, 10^{-1/2}\}$
c)	8	$1 \ \forall \ \ell$
d)	3	$\{1, 10^{-1/4}, 10^{-1/2}\}$



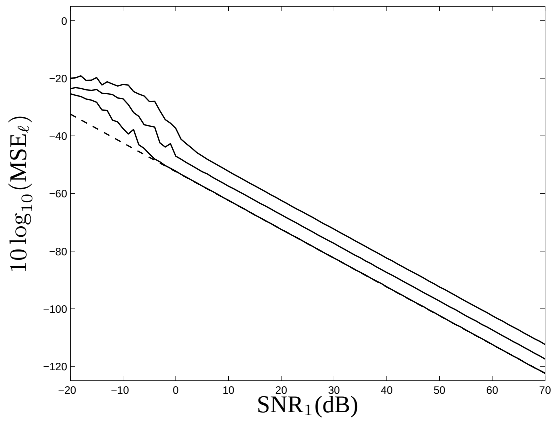
(a) – Eight tones with amplitudes $\alpha_\ell = 1$ for all ℓ.



(b) – Three tones with amplitudes $\alpha_\ell \in \{1,\, 10^{-1/4},\, 10^{-1/2}\}$.



(c) – Eight tones with amplitudes $\alpha_\ell = 1$ for all ℓ.



(d) – Three tones with the amplitudes $\alpha_\ell \in \{1,\, 10^{-1/4},\, 10^{-1/2}\}$.

Figure 1. MSE of the algorithm when estimating the frequencies. The number of data samples is $N = 128$. In 1(a) and 1(b) initialization is performed using the periodogram method. In 1(c) and 1(d) initialization according to $\hat{\omega}_\ell = \omega_\ell + u$, where u is a random variable with uniform distribution in the interval $[-1/(2N), 1/(2N)]$. The individual empiric MSE_ℓ (solid line) as a function of the $\mathrm{SNR}_1 = \alpha_1/(2\sigma^2)$. The asymptotic CRB corresponding to SNR_1 (dashed line) is given as a reference.

cases *c*)–*d*), the threshold is not that distinct as in cases *a*)–*b*), and accurate frequency estimates is obtained for SNRs below 0dB.

In Figures 1(b) and 1(d), it is observed that the proposed algorithm enables to find proper frequency estimates for signals where the amplitudes differ as much as 10dB, without affecting the performance. The threshold effect in Figure 1(b) is not as distinct as in Figure 1(d), making it most probable a result of the particular initialization procedure used.

V. Summary

A real-valued multi-sinewave model has been proposed and the performance of a novel method has been discussed. The four-parameter sinewave fit of [2] [3] has been generalized to a multi-sinewave fit using $3p+1$ parameters. Its performance has been studied by numerical evaluations. The initialization of the algorithm has been briefly discussed and its influence on the performance of the proposed algorithm has been somewhat illustrated.

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