# Cascaded rectangular windows for leakage reduction in spectral analysis

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***Abstract –*** A novel family of windows are proposed for digital spectrum analysis and the main figures of merit compared to those associated with both well-known and state-of-the-art windows available in the literature. The elements of the family, which are obtained by means of successive linear convolutions between elementary rectangular windows, present a very good side-lobe behaviour if a moderately high order of convolution is considered and allow for a strong reduction of the spectral leakage effect at the output of the analysis. The growth of the main-lobe within the spectrum of the proposed windows is acceptable for most of the applications, and the benefits deriving from the rapidly fading side-lobe signature can be appreciated even in presence of the need for a quite high frequency resolution, when the large storage capabilities and computational efficiency provided by modern sampling instrumentation are properly exploited.

## I. Introduction

Windows are a well-known tool which is largely exploited in digital spectral analysis in order to improve the overall performance of the measurement. From a numerical standpoint, a window represents a set of "weighting" values, which are applied to the vector of samples acquired by observing the signal of interest during a *finite* time interval. The concept of "window" is intrinsically introduced by the practical limitation of using an *N*-element vector of samples for estimating the spectral properties of the input signal: since the discrete algorithms that allow for the desired time-to-frequency transformations, such as the Discrete-Time Fourier Transform (DTFT) and/or the Discrete Fourier Transform (DFT, FFT in practice), are described by *generalized* (i.e. "non-finite") series operators, some kind of analytical discussion is needed to the aim of justifying the adoption of these operators in presence of a finite sequence of observations. More precisely, in order to perform the analysis in presence of only *N* experimental samples and, at the same time, preserve the validity of the theory, the following expression can be introduced to mathematically describe the signal $x_w(t)$ whose spectrum is *in practice* being evaluated:

$$x_w(t) = w(t)x(t) \tag{1}$$

$x(t)$ being the *actual* input signal, periodically sampled with step $T_S$ at the time instants

$$t_k = kT_S \qquad k = 0,1,\cdots,N-1 \tag{2}$$

within the observation interval $[0, NT_S]$ and $w(t)$ a *rectangular* window, which allows to extend the experimental information to the entire time-axis if defined as

$$w(t) = \begin{cases} 1 & for \quad 0 \le t < NT_S \\ 0 & otherwise \end{cases} \tag{3}$$

The presence of factor $w(t)$ in Eq. (1) is a source of strong distortion on the spectrum of the input signal. By applying, for example, the DFT operator to the vector of acquired samples, which can be now considered as a *non-finite* sequence since the auxiliary signal $x_w(t)$ is known also outside the experimental observation interval, the following spectrum is evaluated in a discrete sequence of points[1]:

$$X_w(f) = W(f) * X(f) \qquad f \in [0, f_S/2] \tag{4}$$

$W(f)$ being the Fourier transform of window $w(t)$, $X(f)$ the spectrum of the actual input signal $x(t)$ and $f_S = 1/T_S$ the sampling rate. Owing to the "shape" of $W(f)$, *main-lobes* arise in the observed spectrum in correspondence with each spectral tone of the input signal $x(t)$, while *side-lobes* spread around each main-lobe (Fig. 1): the frequency resolution of the analysis is reduced by the first, while the estimated magnitude and

[1] In the present discussion, the Shannon's criterion is considered as fully satisfied. Otherwise, a *periodic extension* with period $f_S$ of function in Eq. (4) should be considered, and the well-known aliasing effects between adjacent spectral images taken into account.

phase of the spectral components of $x(t)$ can be affected (along with the estimation of their frequency location) by strong errors due to the latter (*spectral leakage*). Moreover, the leakage effect can even compromise the detection of weak input tones, leading to the reduction of the instrumental dynamic range.

Many efforts have been made in the last decades in order to identify suitable analytical expressions for $w(t)$ alternative to the basic, "intrinsic" rectangular window described by Eq. (3), which could be capable of improving the spectral analysis overall performance. To this aim, the optimal trade-off between the width of $W(f)$'s main-lobe and the behaviour of side-lobes along the frequency axis has been the goal of many approaches. An exhaustive classification of the windows which can be found in the literature, from pioneering papers up to more recent ones, is available in [1] and [2]. In this paper, a family of windows is proposed, which has been identified with the purpose of decisively reducing the side-lobe spreading by strongly increasing their fall-off rate. The resolution issue has not been ignored however, since the main-lobe widths associated with the proposed windows are still sufficiently narrow to be compatible with most of the applications, especially by considering the large storage capabilities and computational efficiency of modern sampling instrumentation.

## II. Convolution of rectangular windows

In order to minimize the leakage effect affecting the vector spectrum which is observed at the output of the digital analysis, the following family of windows can be considered

$$w^{(R)}(t) = w_1(t) * \cdots * w_i(t) * \cdots * w_R(t) \qquad R = 3,4,\ldots \tag{5}$$

The member $w^{(R)}(t)$ of family (5) is obtained as the linear convolution between $R$ elementary rectangular windows (Eq. (3)), $N_i T_S$ being the width of the $i$th convolution element $w_i(t)$ and $N = (N_1 + \cdots + N_i + \cdots + N_R - R + 1)$ the constraint on parameters $N_i$ to be satisfied in order to match the width of the overall window $w^{(R)}(t)$ with the number of samples obtained from $x(t)$ within the observation interval $[0, NT_S]$. In Fig. 2 the first members of the family (5) are depicted (properly normalized by the peak value), corresponding to the convolution order $R$ = 3,4,5 respectively. According to Eq. (5), the Fourier transform magnitude of window $w^{(R)}(t)$, which is directly related (Eq. (4)) to both the shape of main-lobes and the side-lobe behaviour in the observed spectrum, can be easily expressed by means of the corresponding transform of each contribution to the convolution:

$$\left|\Im\left\{w^{(R)}(t)\right\}(fT_S)\right| = \left|W^{(R)}(fT_S)\right| = \prod_{i=1}^{R} \left|\Im\{w_i(t)\}(fT_S)\right| = \prod_{i=1}^{R} N_i T_S \left|\mathrm{sinc}(N_i fT_S)\right| \tag{6}$$

The Fourier-transformed functions in Eq. (6) have been defined with respect to the *normalized* frequency domain $F = fT_S$ in order to discuss the window spectral properties independently of the sampling step.

The authors recently discussed from a metrological standpoint the overall performance of digital vector spectral analyses involving time-domain filters of Eq. (5)-kind [3,4]. In this paper, the capability of the window family $w^{(R)}(t)$ of dramatically reducing the uncertainty source due to spectral leakage effects is emphasized and discussed. In fact, when a moderately high order of convolution $R$ is taken into account, the corresponding element belonging to the proposed window family (5) reveals an excellent side-lobe behaviour in comparison with that of a large number of well-known and largely applied windows, as well as special-purpose and state-of-the-art windows available in classical and recent literature. In Fig. 3 the normalized spectrum magnitude of $w^{(R)}(t)$ ($R$=3,4,5) is shown, which is analytically described by Eq. (6) as a simple product of *Dirichlet kernels*[2] (i.e. the transform of the elementary rectangular window), each one associated with the $i$th convolution element normalized width $N_i$. The plot shows how the side-lobe signature rapidly fades along the normalized frequency axis, already in correspondence with the first values of the convolution order $R$. For a value $R$ = 5 of the convolution order, the Side-Lobe Fall-Off Ratio (SLFOR) of $W^{(R)}(fT_S)$ achieves a very high value of -30 dB/oct, which can be compared in Tab. 1 with those of some well-known reference windows. Higher rates of SLFOR can be achieved by increasing the order $R$. The SLFOR parameter in fact, which represents the most important figure of merit when strong leakage reduction is requested to the windowing procedure, depends on the analytical continuity of the window and its successive derivatives at the bounds of the observation interval.

[2] In the literature, Dirichlet kernels are commonly used in the framework of synthesizing suitable filtering windows by means, in most of the cases, of linear combinations aimed at side-lobe cancellation effects. The proposed family (5) is obtained by a *direct* multiplication of such kernels, in order to emphasize the side-lobe fading rate associated with the basic function.

The basic rectangular window, which presents a first-order discontinuity, shows a minimum value of the SLFOR= -6 dB/oct. The Bartlett (triangular) window exhibits a SLFOR= -12 dB/oct, since it is continuous at the boundaries but with a non-continuous first-order derivative. From a general standpoint, the window's spectrum presents a SLFOR= $-6(D+1)$, $D$ being the order of the first non-continuous derivative. The proposed family member $w^{(R)}(t)$ is thus characterized by a value of SLFOR= -6$R$ , since this function is continuous up to the ($R$-2)th-order derivative.

The nearby side-lobe behaviour of the window family (5) is very good as well. Windows $w^{(R)}(t)$ present coinciding First and Maximum Side-Lobe Levels (FSLL and MSLL, respectively) which are better than those pertaining to most of the known windows ([1] and Tab. 1) already for $R$ = 4. Once again, an increment of the convolution order turns into a strong improvement on these figures. These parameters should be appreciated by considering that the state-of-the-art for FSLL and MSLL is the so-called Blackman-Harris 4-terms window [1]: this filter is unfortunately characterized by a very low SLFOR (equal to -6 dB/oct), which can strongly reduce the benefits of such a good nearby side-lobe behaviour. An element of the Dolph-Chebyshev family is reported in Tab. 1 for the sake of completeness. The analytical definition of this family would theoretically allow to obtain a window which is characterized by an arbitrary given value of MSLL. This result is however achieved in exchange of a *constant* level of *every* side-lobe (i.e. SLFOR= 0).

Figure 4 shows the $w^{(5)}(t)$ 's normalized spectrum as plotted in Fig. 3, but in comparison with the spectra associated with some well-known windows, which are considered here as significant reference, in order to further validate the discussion above. The spectra are obviously compared by assigning the same width to the windows on the normalized time domain (N=128). As it can be noticed, a moderate $R$ = 5 convolution order leads to a $W^{(5)}(fT_S)$ frequency response not only characterized by a very low MSLL, but which also presents a rapidly fading side-lobe behaviour due to the very high SLFOR, allowing for peaks lower than those pertaining to the Blackman-Harris window spectrum as the normalized frequency increases.

The very good nearby and asymptotic side-lobe behaviour that characterizes the proposed family of windows is a consequence of the direct multiplication between Dirichlet kernels, which describes, in the frequency domain, the successive convolutions at the basis of the family analytical definition (5). The main drawback of this technique is thus the growth of the main-lobe in the window frequency response. In Tab. 1 the 3-dB and 6-dB Band-Widths (BW), which measure the width of the main-lobe at the half-power and at the 0.5-attenuation points respectively, are reported in frequency *bins* (1 bin = $f_S/N$ is the frequency interval $[0, f_S]$ quantization step adopted by a DFT-like analysis) for both the reference windows and the $R$=3,4,5-order convolution windows. These parameters increase, as expected, with the order of convolution and actually reveal a worse performance in terms of resolution. Nevertheless, the increment of the main-lobe width is still acceptable when $R$, instead, has already achieved values that allow for very high performance in terms of leakage reduction. Moreover, it is essential to notice that the 3- and 6-BWs depend, for a given value of the sampling rate, on the number $N$ of signal samples acquired within the observation interval. Modern sampling instrumentation, even at the entry-level, provides storage capabilities that allow, in the practice, for very small values of the fundamental bin quantity and make the resolution issue a mild constraint in the choice of the most suitable window. Thus, apart from those special applications that require either ultra-resolution performance or the investigation on the input signals within very short observation intervals, the RAM memory of modern spectrum analyzers, together with the always increasing speed of computation which allows to manage large amounts of data, can be exploited in order to successfully apply the proposed windows in most of the cases of interest. In the next section, numerical examples will be provided, which show the overall performance achievable by adopting a convolution window of kind (5) for the digital spectrum analysis of a high-dynamic signal, even in presence of the need for quite high resolution.

## III. Application examples

In the following, an *intermodulation distortion* measurement, performed through a digital spectrum analysis, will be considered. Such an investigation represents a fundamental test for the experimental characterization of the in-band distortion generated at the output of a non-linear system (e.g. an RF amplifier for highly linear applications). In the framework of the present example, the signal analyzed is the response of a non-linear system to a two-sinusoidal tone input signal. Under such excitation, in-band *third-order intermodulation products* are generated at the output of the system at the frequencies $f_{12} = 2f_1 - f_2$ and $f_{21} = 2f_2 - f_1$ due to the non-linear effects, $(f_1, f_2)$ being the spectral position of the two excitation tones. The frequency displacement $\Delta f = f_2 - f_1$ between these inputs is usually chosen very small with respect to the bandwidth of the system, mainly to make the response independent of the selectivity of the system. The values $f_0$ = 1.5 MHz and $\Delta f$ = (1000+π) Hz have been considered for the generation of $f_1 = f_0 - \Delta f / 2$, $f_2 = f_0 + \Delta f / 2$. The π contribution in $\Delta f$ is justified by the need of generating two *non-commensurable* input frequencies in order to have a quasi-periodic regime within the non-linear system. The obtained intermodulation products are characterized by a -70 dB level under the tones $(f_1, f_2)$ at the output of the system.

Thus, the problem of detecting and correctly estimating such weak spectral components in presence of nearby high-level tones introduces the need for a very-low leakage measurement procedure. In addition, an high frequency resolution is needed in order to resolve the signal spectral content. Figure 5-a shows the response of the FFT analysis performed by considering a sampling rate $f_S$ = 5 MS/sec, $N = 10^5$ signal samples (1 bin = 50 Hz) and a basic, rectangular window. The frequency resolution is clearly adequate to resolve the main tones at $(f_1, f_2)$, but the low SLFOR of the rectangular window strongly reduces the dynamic range of the instrument and practically generates a "leakage floor" that prevents the detection of the third-order intermodulation products. The exploitation of the Hamming window, with the same SLFOR but a much lower MSLL allows for a higher dynamic (Fig. 5-b), but the leakage is still unacceptable. Figure 5-c shows the results of the analysis performed by using the Modified Bartlett-Hanning (MBH) family of windows [5,6], which is defined as a linear combination between the triangular and the $\cos^2(t)$ -based windows. This analytical definition allows for an highly flexible family, whose elements can be chosen by privileging the resolution performance or, instead, the side-lobe behaviour. In Tab. 1 the figures of merit of the MBH-element (α=1.87) with the narrowest main-lobe and those associated with the element (α=.6404) characterized by the best side-lobe behaviour are listed, respectively. The use of the latter allows for a mild detection of the weak tones within the signal spectrum, but the leakage prevents from clearly resolving and estimating these spectral components. Finally, in Fig. 5-d the tones under investigation are correctly estimated by using the $R$=5-order convolution window. The choice of the value $N = 10^5$ has allowed to adequately reduce the main-lobe widths even in presence of a very narrow $\Delta f$ , and make negligible the resolution issue. Instead, the high-performance of the convolution window in terms of side-lobe figures has led to a decisive reduction of the leakage effect and a successful measurement.

It is worth to notice that even the entry-level sampling instrumentation provides nowadays the capability of acquiring, storing and processing the number of samples considered in this example. In practice, $N$ can achieve the value of many million points in average-cost spectrum analyzers, or by means of non-expensive plug-ins. Thus, modern instrumentation is capable of managing amounts of data which are large enough in order to implement the use of moderate-order convolution windows of family (5) and exploit the benefits of the leakage reduction capabilities discussed above, even in presence of the need for a quite high frequency resolution.

## IV. Conclusions

The main figures of merit of a novel family of windows have been presented and compared to those of several reference windows. The proposed family is obtained by means of successive linear convolutions between elementary rectangular windows. The resulting frequency response, which can be described in terms of products between Dirichlet kernels, exhibits an excellent nearby as well as asymptotic side-lobe behaviour, better than those of state-of-the-art windows available in the literature if a moderately high order of convolution is considered. The growth of the main-lobe width, which increases with the order of the convolution, is acceptable in order to keep the resolution performance adequate enough for most of the applications. This is further confirmed by considering the large storage capabilities and high computational efficiency provided by modern sampling instrumentation, which allows for the processing of a large amount of signal samples. For these reasons, the adoption of the proposed family in the framework of digital spectrum analysis represents a suitable choice in order to obtain a strong reduction of the leakage effect. Numerical examples have been provided, which validate the discussion and show the benefits achievable by the exploitation of the proposed filters also in presence of the need for a quite high frequency resolution.

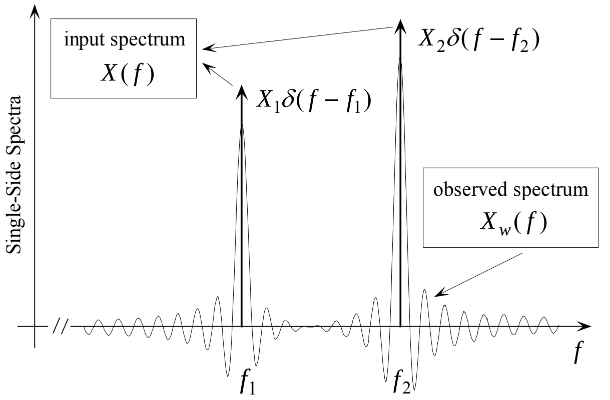


Fig. 1 - Distortion on the spectrum of a two-tone input signal with harmonics ($X_1$,$X_2$) at the output of the digital analysis, due to the presence of the rectangular observation window in Eq. (1).

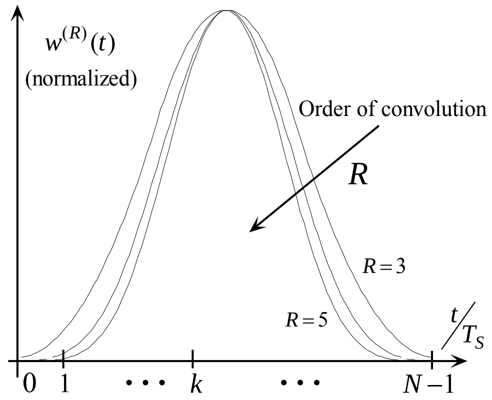


Fig. 2 – Normalized-amplitude plot of the windows belonging to the proposed family for the convolution orders $R$ = 3,4,5.

| Window | 3-dB BW [bins] | 6-dB BW [bins] | FSLL [dB] | MSLL [dB] | SLFOR [dB/oct] |
|---|---|---|---|---|---|
| Rectangular | 0.89 | 1.21 | -13.3 | -13.3 | -6 |
| Bartlett | 1.28 | 1.78 | -26.5 | -26.5 | -12 |
| Hanning (Hann) | 1.44 | 2.00 | -31.6 | -31.6 | -18 |
| Hamming | 1.30 | 1.81 | -45.4 | -42.6 | -6 |
| Modified Bartlett-Hanning (α=1.87) [6] | 1.08 | 1.48 | -24.3 | -16.5 | -12 |
| Modified Bartlett-Hanning (α=0.6404) | 1.39 | 1.93 | -36.9 | -36.9 | -12 |
| Blackman-Harris 4-terms | 1.90 | 2.72 | -100.0 | -92.1 | -6 |
| Dolph-Chebyshev (α=4.0) | 1.65 | 2.31 | -80.0 | -80.0 | 0 |
| Cascaded Rectangular 3rd-order | 1.56 | 2.17 | -39.8 | -39.8 | -18 |
| Cascaded Rectangular 4th-order | 1.76 | 2.47 | -53.0 | -53.0 | -24 |
| Cascaded Rectangular 5th-order | 2.01 | 2.81 | -66.1 | -66.1 | -30 |

Tab. 1 – Main figures of merit of some windows available in the literature, compared to those of R=3,4,5-members of the proposed family.

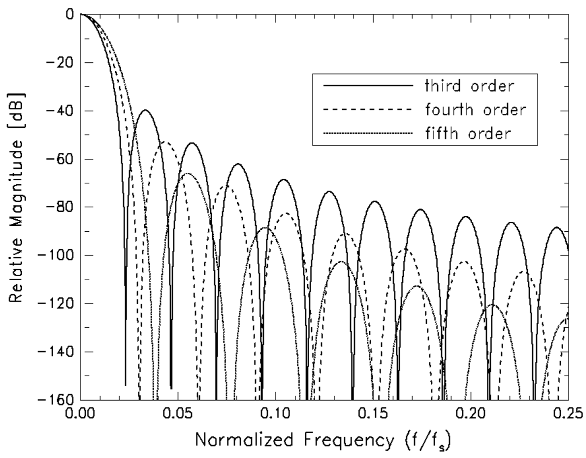


Fig. 3 – Spectra of the proposed windows for the convolution orders R=3,4,5.

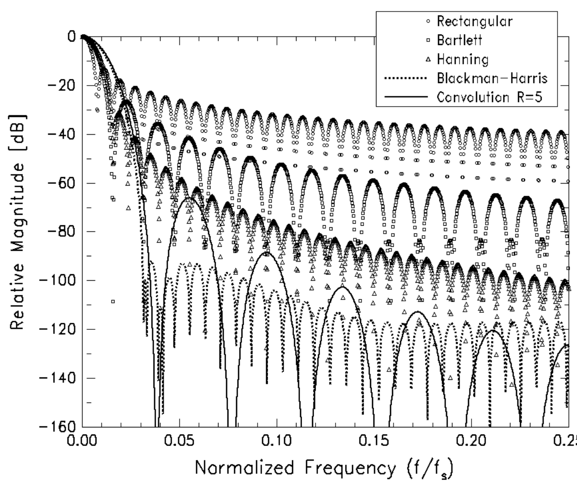


Fig. 4 – Spectrum of $w^{(5)}(t)$ compared to the lobe behaviour of some reference windows considered in Tab. 1.

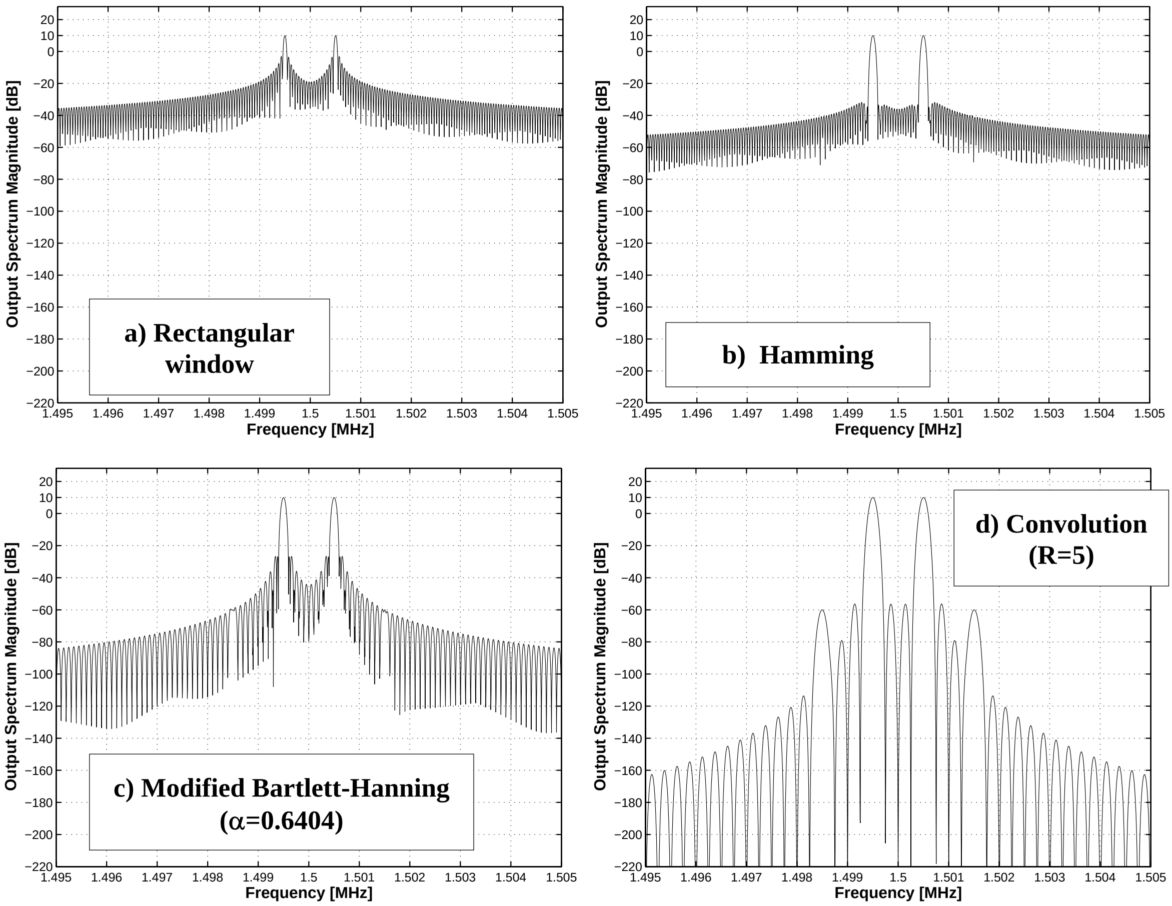


Fig. 5 – Digital spectrum analysis on the third-order intermodulation products at the output of a non-linear system, performed by exploiting the a) rectangular, b) hamming, c) MBH with best side-lobe figures and d) fifth-order convolution windows. (Signal parameters: $f_0$ = 1.5 MHz, $\Delta f$ = (1000+$\pi$) Hz, level of third-order products = -70 dBc. FFT: $N = 10^5$ samples with $f_S$ = 5 MS/sec).

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