Calculation of Expanded Uncertainties without Knowledge of Coverage Factor

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Abstract- The main purpose of the paper is to present a new fully deterministic, numerical approach to calculation of coverage interval at certain coverage confidence. The main crucial aspects for this method are revealed and the accuracy of this method is presented as well. A brief comparison to others methods, analytical and numerical based on Monte Carlo methods and also so widely applied and recommended in GUM law of propagation of uncertainties are described. Several examples are presented.

I. Introduction

It is very well known that no measurement is exact. There is a certain uncertainty at certain degree of confidence, which is always associated to measurement. It is important to specify the uncertainty of the measurement along with the result of the measurement. Estimation of uncertainty and the way of uncertainty presentation numerical and graphical forms are in the scope of interest of everyone who is involved in any measurements, as otherwise the knowledge is incomplete. Measurements for control systems, for calibration process required as precisely as possible [1]. There is a number of different sources for measurement error, such as the measurement method, imperfection of instrumentation used and measurement environment. Therefore, it is important to identify the most significant source of errors so that less significant ones may be neglected. The first step to take in the reduction of errors is by understanding them.

II. Classification of uncertainty evaluation methods

There are a lot of efforts done during year to find the solution for combining random and systematic errors [2,3]. The GUM [4] elaborated and accepted by 7 organisations was a milestone in classification of methods of uncertainty evaluation. The methods may be carried from different point of view like:

- analytical methods [GUM Subclause G.1.5]
- method with an arbitrary value of coverage factor k ($k = 2$ or 3)
- approximation method [5]
- numerical approach based on probability density function; here three kinds of methods are pointed: the method with a PDF generation based on bootstrap functions [6], Monte Carlo method of calculation of PDF [7] and the convolution method based on FFT and IFFT algorithms [8].

Propagation of distributions is a generalization of the approach predominantly advocated in the GUM, in that it works with richer information than that conveyed by best estimates and the associated standard uncertainties alone.

The approximation method of k-factor evaluation is based on consideration of which components have a major contribution. The value of coverage factor depends on the ratio r of the dominant contribution $c_i\, u(x_i)$, having the rectangular distribution, and the geometric sum of the rest contributions:

$$r = \frac{|c_i u(x_i)|}{\sqrt{u_c^2 - c_i^2 u^2(x_i)}} \tag{1}$$

The unknown PDF of a measurand is approximated by means of one out of three distributions: normal, trapezoidal or rectangular. The following formulation of coverage factor is applied:

$$k = k_N = 2 \qquad for \qquad 0 \le r < 1$$
$$k = k_T = \sqrt{\frac{3}{r^2+1}}\left(1 + r - 2\sqrt{r(1-p)}\right) \qquad for \qquad 1 \le r \le 10 \tag{2}$$
$$k = k_R = \sqrt{3}p \qquad for \qquad r > 10$$

The approximation method for calculation of expanded uncertainty in calibration of RTD sensors at low temperature is described in [9]. The trapezoidal shape was chosen as the most appropriate because two components have the most significant contributions.

Monte Carlo method is a probabilistic one and is similar to the method with bootstrap functions. The effect depends on a number of randomly generated points.
The method developed by authors is also based on PDF function, but the fast algorithms from DSP: FFT and IFFT have been applied.

III. The convolution method

The proposed approach to calculation of expanded uncertainty is based on calculation of convolutions of all components with their sensitivity factors. The authors have developed a software of which algorithm is presented in Fig. 1. The software is available at www.metrol.p.lodz.pl and any user can perform calculations via the Internet.
The software handles the following probability distribution functions: uniform and t-Student distribution. The normal distribution is related to t-Student by very large number of degree of freedom. Trapezoidal and triangular shapes of probability distribution functions are achieved from two uniform distributions. All calculations are performed on digital representations. The piece of software is based on calculations: the convolution of all components, the cumulative distribution as an integration of achieved convolution and finally the expanded uncertainty *U* at a desired confidence level. Confidence level *p* can be chosen by software user as any value from the range (0÷1). Finally, the *k*-factor is calculated just a quotient of the expanded and combined uncertainties.
The most important element to achieve appropriate accuracy of calculation of expanded uncertainty is to sample input probability distribution functions with a proper number of discreet points. The number of this points comes from consideration of how accurate the coverage factor we would like to have, although it is not used for calculation of expanded uncertainty at all. It helps only to find appropriate number of samples and consequently distance between samples of probability distribution functions and cumulative distribution.
Let's assume accuracy of *k*-factor calculation up to 5 significant digits, for example: 2.1234 (0,005 %). In this case the number of samples $N_{\max}$ needed for fast Fourier algorithm is $N_{\max} = 2^{14}$ for $p = 0.95$ or $N_{\max} = 2^{15}$ for p=0.99. After that is possible to calculate number of samples per a unit of error - Equ. (3) in which denominator represents all components contribution to expanded uncertainty.

$$N_{UNIT} = Ent\left(\frac{N_{\max}}{2\left(\sum_{i=1}^{i=N} c_i u_{Ai} t_{\max i} + \sum_{i=1}^{i=N} c_i u_{Bi}\sqrt{3}\right)}\right) \tag{3}$$

where parameter $t_{\max i}$ is given in Table 1.
For t-Student distribution coefficient of $t_{\max i}$ were calculated based on experimental method to achieve the assumed accuracy of *k*-factor calculation.

Table 1. Maximum values of $t_{\max}$ parameter for t-Student and Gaussian distribution

p, ν	2	3	4	5	6	7	8	9	10	12	15	20	30	50	>50
0.95	90	32.0	20.0	16.0	13.6	12.0	11.2	10.5	10.0	9.0	8.2	7.4	6.6	6.1	6.0
0.99	125	41.0	25.5	20.0	17.0	15.5	14.3	13.4	12.4	11.2	10.1	9.0	8.0	7.3	7.0

Number of samples for uniform distributions

$$N_{Ri} = Ent\left(2\sqrt{3}N_{UNIT} c_i u_{Bi}\right) \tag{4}$$

and the value of probability density

$$g_{Ri} = \frac{1}{N_{Ri}} \qquad (5)$$

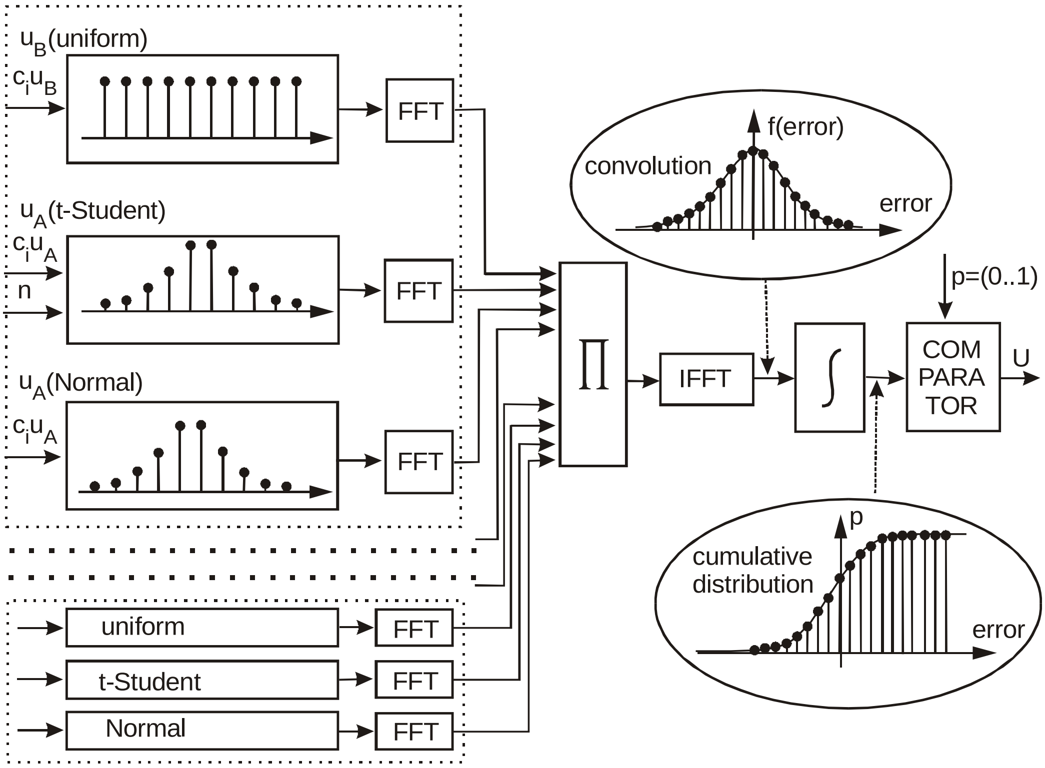


Figure 1. Block diagram of expanded uncertainty calculation. FFT – Fast Fourier Transform, Π – product, IFFT – Inverse Fast Fourier Transform

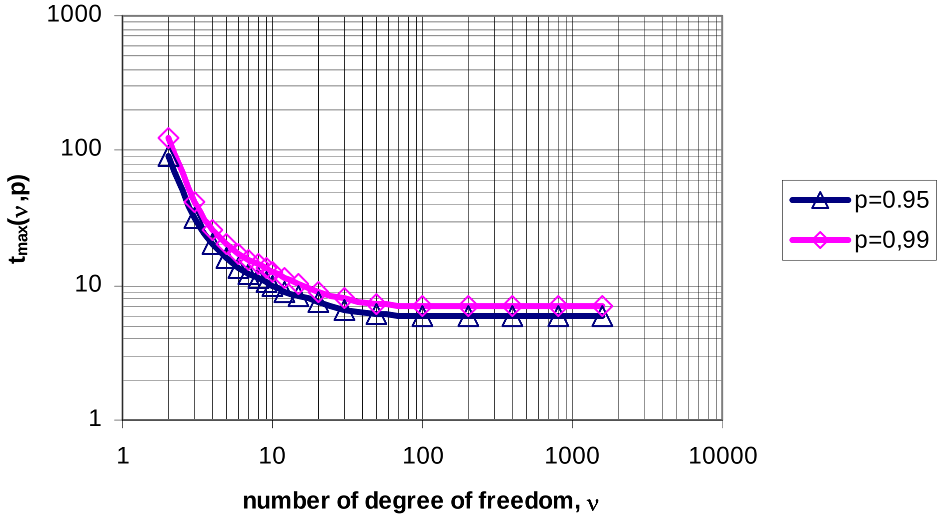


Figure 2. The relation of $t_{max}(\nu, p)$ coefficient vs. degrees of freedom for t-Student distribution, which allows calculate number of samples

The number of samples of t-Student distribution or Gauss distribution

$$N_{Si} = Ent\left(2N_{UNIT} c_i u_{Ai} t_{\max i}\right) \qquad (6)$$

Probability density function is

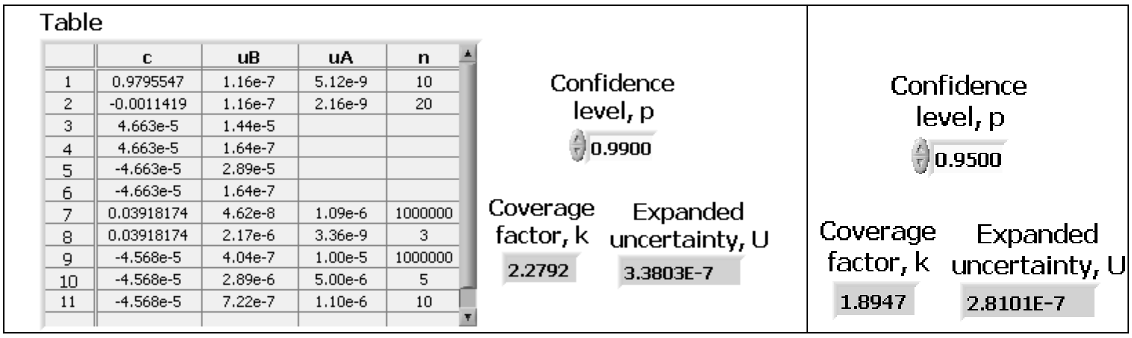
$$g_{Sij} = G(\nu_i)\frac{\Delta t_i}{\left(1+\frac{(j+0.5)^2\Delta t_i^2}{\nu_i}\right)^{\frac{\nu_i+1}{2}}} \quad \text{for } j=-\frac{N_{si}}{2} \div \frac{N_{si}}{2}-1 \tag{7}$$

where:

$$G(\nu_i)\frac{\Gamma\left(\frac{\nu_i+1}{2}\right)}{\sqrt{\pi\nu_i}\,\Gamma\left(\frac{\nu_i}{2}\right)} \qquad \Delta t_i = \frac{N_{UNIT}}{c_i u_{Ai}} \tag{8}$$

The cumulative distribution function we achieve as a sum of convoluted samples.

Expanded uncertainty U is calculated for $\alpha = \frac{1}{2}(1-p)$.



Table

	c	uB	uA	n
1	0.9795547	1.16e-7	5.12e-9	10
2	-0.0011419	1.16e-7	2.16e-9	20
3	4.663e-5	1.44e-5		
4	4.663e-5	1.64e-7		
5	-4.663e-5	2.89e-5		
6	-4.663e-5	1.64e-7		
7	0.03918174	4.62e-8	1.09e-6	1000000
8	0.03918174	2.17e-6	3.36e-9	3
9	-4.568e-5	4.04e-7	1.00e-5	1000000
10	-4.568e-5	2.89e-6	5.00e-6	5
11	-4.568e-5	7.22e-7	1.10e-6	10

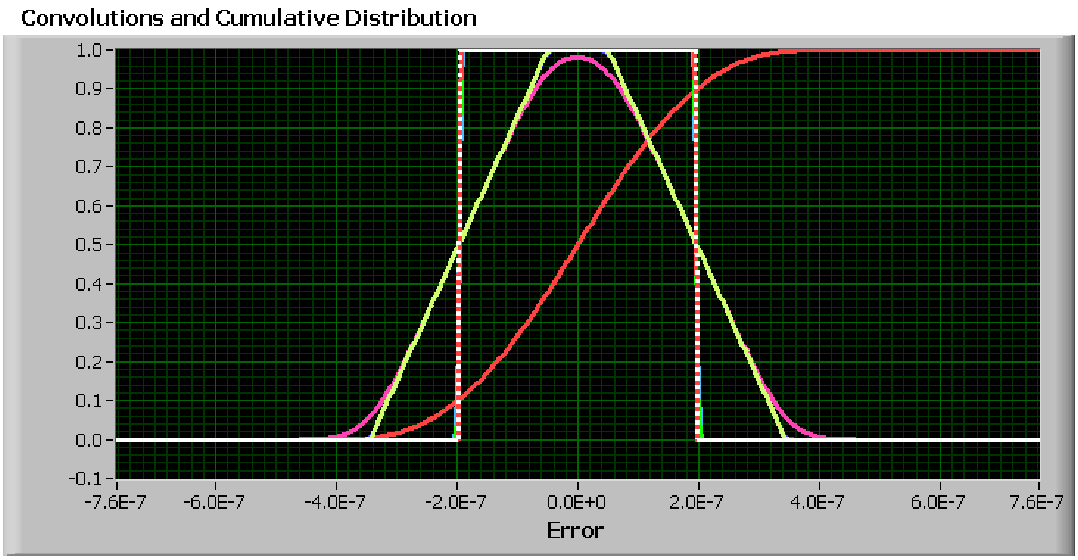


Figure 3. Image of the screen for calculations performed by the convolution method at the levels of confidence $p = 0.99$ and $p = 0.95$.

IV. Comparison of methods

Two methods of uncertainty calculation: with arbitrary value of k and the approximated method were compared using the new convolution method as a standard. The comparison of results is done for the following most popular probability distribution functions:

- one normal and one rectangular distribution (typical for combining type A and type B uncertainties)
- one t-Student for 2 degree of freedom and one rectangular distribution
- set 1 to 10 of rectangular probability distribution functions.

The errors of calculated k-factor are presented in the Fig. 3, Fig. 4 and Fig. 5.

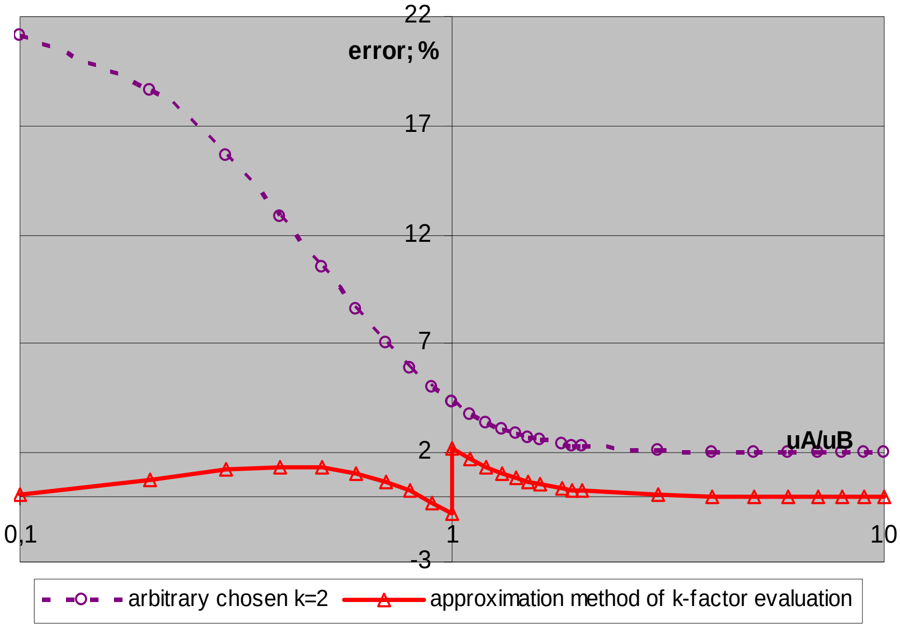


Figure 3. The error of *k*-factor evaluation for Gaussian and rectangular distributions for $p = 0.95$

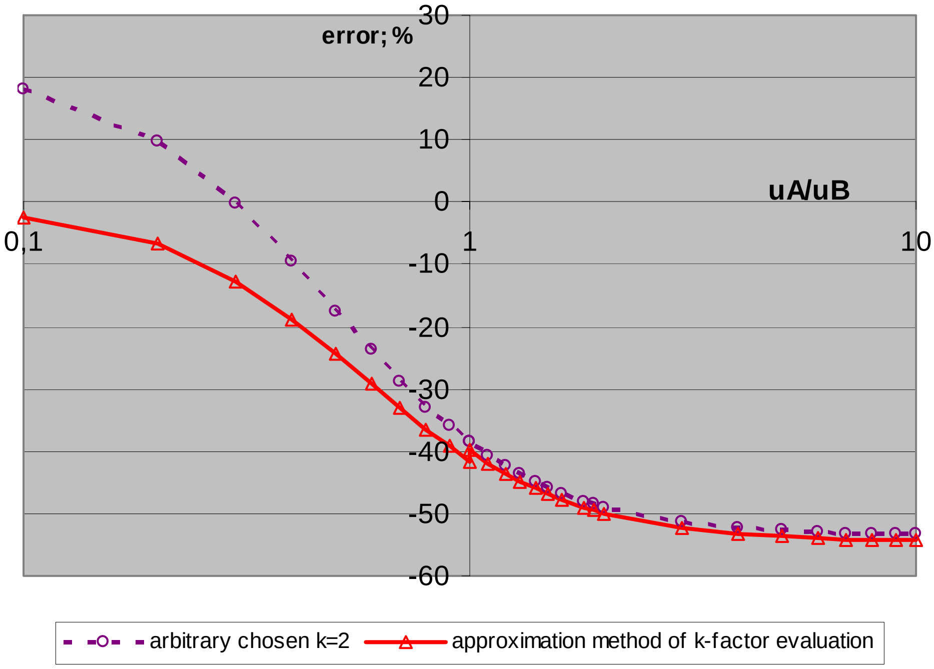


Figure 4. The error of *k*-factor evaluation for t-Student and rectangular distributions for $p = 0.95$

V. Conclusions

The problem of calculation of expanded uncertainty is still under deliberation. The final conclusion of the paper denotes that new information technology supports calculation of uncertainties so much, that nowadays each personal computer and each instrument can be equipped in modules supporting calculation of uncertainties based on direct method coming from definition of uncertainty and based on convolution [10]. The presented algorithm is dedicated to all calibration laboratories and everyone who would like to calculate expanded uncertainty without considering the shape of the final convoluted probability density function.

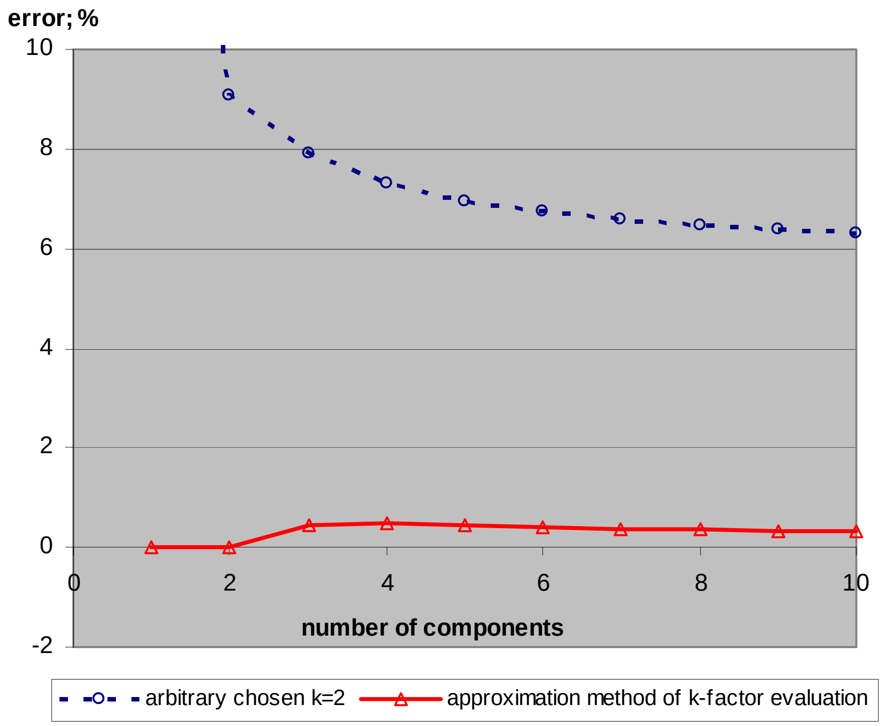


Figure 5. The error of *k*-factor vs. number of rectangular components contributing to expanded uncertainties calculations performed for $p = 0.95$

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