Fourier Spectrum of D/A Outputs with Non-uniformly Sampled Data and Time-Varying Clocks

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Summary

In this paper, we investigate problems of D/A converters with non-uniformly sampled input data, and/or time-varying clock sources. The input digital data (which are stored in the memory to be read out and sent to a D/A converter) were obtained by sampling an analog waveform at non-uniform sampling intervals. (Quantization of data can also be considered as a form of non-uniform sampling.) Recently, there is available a new clocking system with a very fine time resolution and is capable of adjusting the clock period in a sample-to-sample basis. It is, therefore, interesting to consider the following question**. "Given that the timing offset of each data sample is known, would it be beneficial to use this offset to adjust the read-out timing of the D/A converter?"** To answer this question, we consider the following five different models: $f_1(t) = \Sigma_n x(nT)g(t-nT)$, $f_2(t) = \Sigma_n x(t_n)g(t-nT)$, $f_3(t) = \Sigma_n x(nT)g(t-t_n)$, $f_4(t) = \Sigma_n x(t_n)g(t-t_n)$, *and* $f_5(t) = \Sigma_n x(t_n)g_n(t-t_n)$ where $x(.)$ is the input analog signal, $g(.)$ is the basic output pulse waveform of the D/A converter, T is the nominal sampling period and t_n is the n-th sampling time instance (for uniform sampling $t_n=nT$). Closed form expressions for the Fourier transform of the output signals for each model are derived. We also discuss some potential practical applications**. Key Words: D/A Converters, Non-uniform Sampling, Fourier Spectrum**.

1. Introduction

A typical model for the operation of a digital to analog (D/A) conversion is shown in Figure 1.

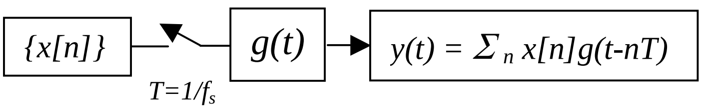

Figure 1

Where $\{x[n]\}$ is a sequence of digital numbers (digital signal), f_s is the clock frequency, and $g(t)$ is a basic pulse waveform determined by the D/A converter. At each clock tick, the converter outputs a basic pulse waveform multiplied by the input digital signal. For example, at $t=0$, the converter outputs $x[0]g(t)$, and at $t = kT$, the converter outputs $x[k]g(t-kT)$, where $T=1/f_s$ is the clock period. It is clear that the output waveform $y(t)$ is given by $\Sigma_n x[n]g(t-nT)$. The output waveforms of a traditional D/A converter with two simple basic pulse waveforms, $g(t)$, and uniform sampled digital data are shown in Figure 2.

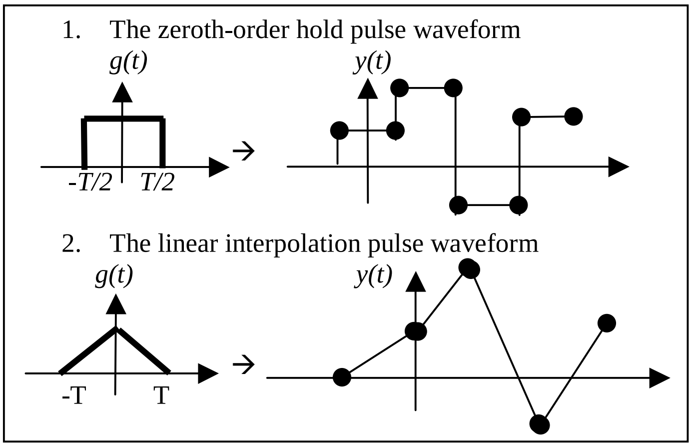

Figure 2

The mathematical model for this kind of standard D/A operation is given by

$$f_1(t) = \Sigma_n x(nT)g(t-nT) \quad (1)$$

The term $x(nT)$ indicates that the digital data, $x[n]$, is coming from uniform sampling of a continuous time signal $x(t)$ with the sampling period T. The term $g(t-nT)$ indicates that the D/A clock is of constant frequency, and taking digital data at a regular time interval T. If the digital data $x[n]$ are obtained from non-uniformly sampling the continuous time signal $x(t)$, then the model is [2]

$$f_2(t) = \Sigma_n x(t_n)g(t-nT) \quad (2)$$

where t_n are actual sampling time instances. If the D/A clock period can change from sample to sample, then the model of the output signal with uniformly sampled data is given by [3]

$$f_3(t) = \Sigma_n x(nT)g(t-t_n) \quad (3)$$

If the digital data $x[n]$ are obtained from non-uniformly sampling the continuous time signal $x(t)$ and that the D/A clock are adjusted to match the corresponding sampling time instances, then the model becomes [2]

$$f_4(t) = \Sigma_n x(t_n)g(t-t_n) \quad (4)$$

Finally, the most general model is

$$f_5(t) = \Sigma_n x(t_n)g_n(t-t_n) \quad (5)$$

Noticed that in this model, the basic output pulse waveforms, $g_n(.)$, are changing from sample to sample. Equations *(5)* models the output signal of a D/A converter with conventional zeroth-order hold circuit with non-uniform sampled data and a time varying clock. Two possible D/A converter output waveforms are sketched in Figure 3 (a)-(b). Equation *(3)* or *(4)* can be used to model Figure 3(a), and Figure 3(b) can be modeled by Equation *(5)*.

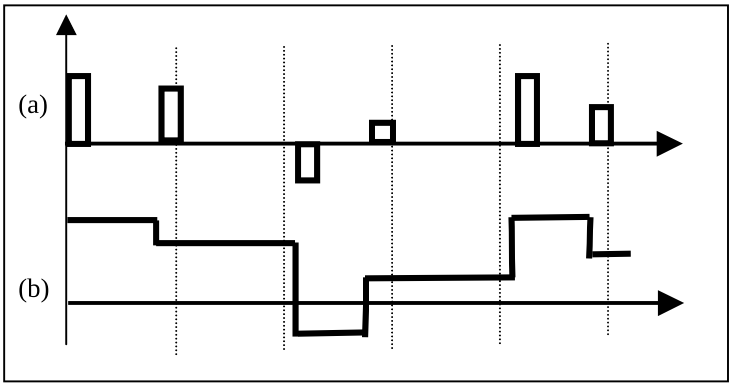

Figure 3

For all models considered in this paper, the non-uniform timing instance is assumed to have a periodic structure with the period M, That is

$$t_n = nT + \Delta_n \qquad (6)$$

where T is the nominal clocking (or sampling) period, and Δ_n is a periodic sequence with period M. This model covers many practical situations and its motivation was discussed in detail in [1] and [2]. Let $n= kM + m$, then

$$t_n = (kM + m)T + \Delta_{kM+m} = kMT + mT + \Delta_m$$
$$= kMT + mT + r_mT \qquad (7)$$

where $r_m = \Delta_m /T$ is the timing offset measured in percentage of the nominal sampling period.

2. Fourier spectrum of traditional D/A outputs

With the output waveform $f_1(t) = \Sigma_n x(nT)g(t-nT)$ as shown in Equation *(1)* we have the Fourier transform, $F_1(\Omega)$, of $f_1(t)$, given by

$$F_1(\Omega) = \int f_1(t)e^{-j\Omega t}\, dt = \int[\Sigma_n x(nT)g(t-nT)]\, e^{-j\Omega t}\, dt$$
$$= G(\Omega)\, [\Sigma_n x(nT)\, e^{-j\Omega nt}]$$
$$= G(\Omega)X(\Omega T) \qquad (8)$$

Where $X(\Omega T)$ is the discrete time Fourier transform (DTFT), $X(\omega)$, of $\{x[n]\}$ evaluated at $\omega=\Omega T$, and $G(\Omega)$ is the Fourier transform of $g(t)$. A typical plot of $X(\Omega T)$, $G(\Omega)$, and $F_1(\Omega)$ are shown below in Figure 4. Usually, one uses an analog filter, after the D/A converter, to remove the energy beyond half of the sampling frequency.

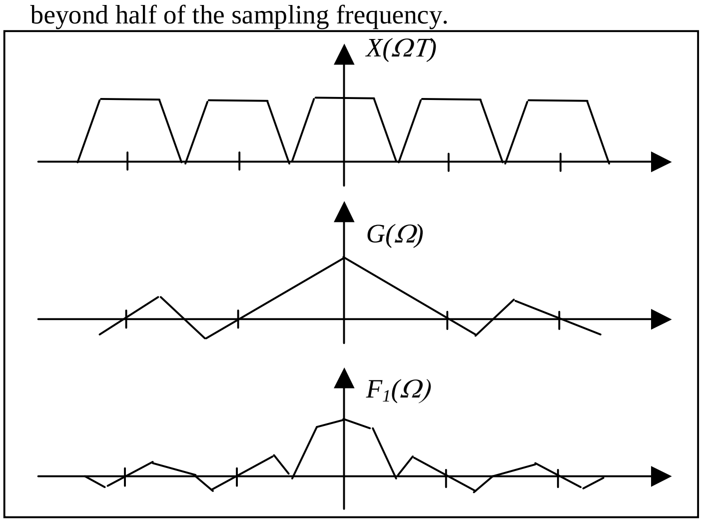

Figure 4

Here, we are interested in how much attenuation $G(\Omega)$ can achieve for frequencies above the sampling frequency for the two cases mentioned earlier. For the zeroth-order hold pulse waveform, the $G(\Omega)$ is given by

$$T\, [Sin\, (\Omega T/2) / (\Omega T /2)] \qquad (9)$$

and for the linear interpolation pulse waveform, the $G(\Omega)$ is given by

$$T^2\, [Sin\, (\Omega T/2) / (\Omega T /2)]^{2} \qquad (10)$$

In the following figure (Figure 5), we plot Equation *(9)* (we normalized the sampling frequency to be one, and hence $T=1$) in dB versus frequency.

Attenuation for zeroth-order hold pulse waveform

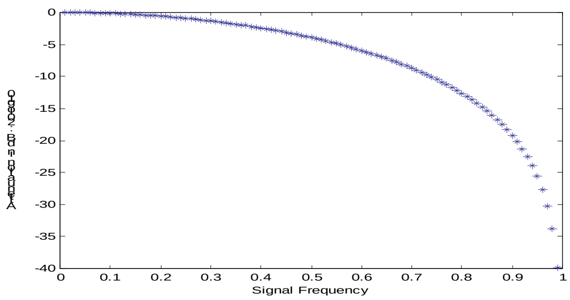

Figure 5

It is seen that, for the zeroth-order hold pulse waveform, the signal at *40%* of the sampling frequency will suffer about *–2.42dB* attenuation. For the signal at *60%* of the sampling frequency (which corresponds to the first image of the original signal at *40%* of the sampling frequency), we will have only about *–5.95 dB* attenuation. At *90%* of the sampling frequency (which corresponds to the first image of the original signal at *10%* of the sampling frequency), we will have about *-20 dB* attenuation. At *99%* of the sampling frequency (which corresponds to first image of the original signal at *1%* of the sampling frequency), we will achieve about *40 dB* attenuation; For the linear interpolation pulse waveform, the amount of attenuation is twice that of the one shown in Figure 5, because equation *(10)* is the square of equation *(9)*. That is, the signal at *40%* of the sampling frequency will suffer about *–4.84 dB* attenuation. For signal at *60%* of the sampling frequency, we will have about *–11.9 dB* attenuation. At *90%* of the sampling frequency, we will have about *-40 dB* attenuation. At 99% of the sampling frequency,we will achieve about -80 dB attenuation.

3. Fourier spectrum of D/A outputs with non-uniformly sampled input signal [2]

If the input digital data $x[n]$ are obtained by non-uniformly sampling a continuous time signal $x(t)$, then the output waveform $f_2(t)$ is given by Equation *(2)*, and we have the Fourier transform, $F_2(\Omega)$, of $f_2(t)$ given by

$$F_2(\Omega) = \int f_2(t)e^{-j\Omega t}\, dt = \int[\Sigma_n x(t_n)g(t-nT)]\, e^{-j\Omega t}\, dt$$
$$= \int[\Sigma_{m=0,(M-1)} \Sigma_k\, x(kMT+mT+r_mT)g(t-kMT-mT)]\, e^{-j\Omega t}\, dt$$
$$= G(\Omega)[\Sigma_{m=0,(M-1)} \Sigma_k\, x(kMT+mT+r_mT)\, e^{-j\Omega kMT}\, e^{-j\Omega mT}]$$
$$= G(\Omega)\Sigma_{m=0,(M-1)}\, [(1/2\pi) \int X(\omega)\, e^{j(\omega-\Omega)mT}\, \Sigma_k\, e^{jk(\omega-\Omega)MT}\, d\omega\, exp(j\omega r_mT)]$$

By applying the following identity [4]

$$\Sigma_k\, e^{jk(\omega-\Omega)MT} = (2\pi/MT)\, \Sigma_k\, \delta(\omega-\Omega+k(2\pi/MT)) \qquad (11)$$

we have

$$F_2(\Omega) = G(\Omega)(1/MT)\Sigma_{m=0,(M-1)}\Sigma_k X(\Omega\text{-}k[2\pi/MT])$$
$$exp(jr_m[\Omega T\text{-}k\{2\pi/M\}])\ e^{-jkm(2\pi/M)}$$
$$= G(\Omega)(1/T)\Sigma_k A_{2,k}(\Omega)\ X(\Omega\text{-}k[2\pi/MT]) \qquad (12)$$

where

$$A_{2,k}(\Omega) = (1/M)\Sigma_{m=0,(M-1)} exp(jr_m[\Omega T\text{-}k\{2\pi/M\}])e^{-jkm(2\pi/M)} \qquad (13)$$

Equations *(12)* and *(13)* completely characterize the Fourier transform $F_2(\Omega)$. It is noted that $A_{2,k}(\Omega)$ is a function of both k and Ω. If $x(t)$ is a sinusoidal $exp(j\Omega_o t)$, which has a Fourier transform $X(\Omega) = 2\pi\delta(\Omega\text{-}\Omega_o)$. Then, Equation *(12)* becomes

$$F_2(\Omega) = G(\Omega)(1/T)\Sigma_k A_{2,k}(\Omega)2\pi\delta(\Omega\text{-}\Omega_o - k[2\pi/MT])$$
$$= G(\Omega)(1/T)\Sigma_k A_{2,k}(\Omega_o T + k[2\pi/MT])2\pi\delta(\Omega\text{-}\Omega_o - k[2\pi/MT])$$

where

$$A_{2,k}(\Omega_o T + k[2\pi/MT]) = (1/M)\Sigma_{m=0,(M-1)}\ exp(jr_m\Omega_o T)e^{-jkm(2\pi/M)}$$

It is noted that $A_{2,k}(\Omega_o T + k[2\pi/MT])$ is the DFT (discrete Fourier transform) of $\{exp(jr_m\Omega_o T)\}$, $m = 0,1,2, \ldots(M\text{-}1)$. Using Parseval theorem, one can easily calculate the signal to noise ratio, SNR, as follows:

$$SNR = 10\ log\ (|A_{20}(\Omega_o)|^2 / [1\text{-}|A_{20}(\Omega_o)|^2]\ ) \qquad (14)$$

where

$$A_{2,0}(\Omega_o) = (1/M)\Sigma_{m=0,(M-1)}\ exp(jr_m\Omega_o T) \qquad (15)$$

A typical plot of $F_2(\Omega)\ /G(\Omega)$ for a sinusoidal input is shown in Figure 6.

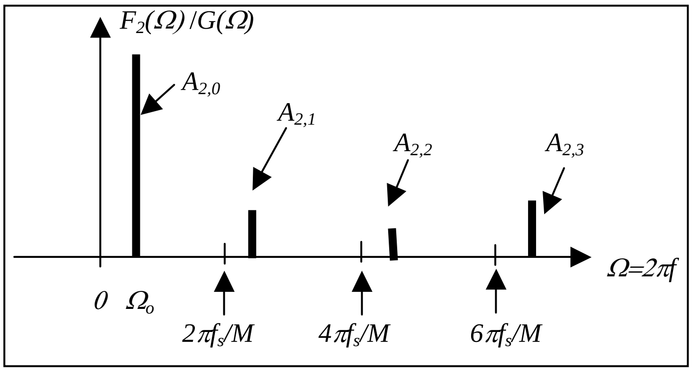

Figure 6

4. Fourier spectrum of D/A outputs with time-varying clock [3]

If the D/A clock period can change from sample to sample, then the model of the output signal with uniformly sampled data is given by $f_3(t) = \Sigma_n x(nT)g(t\text{-}t_n)$, and we have

$$F_3(\Omega) = \int f_3(t)e^{-j\Omega t}\,dt = \int[\Sigma_n x(nT)g(t\text{-}t_n)]\ e^{-j\Omega t}\,dt$$
$$= \int[\Sigma_{m=0,(M-1)}\ \Sigma_k\ x(kMT+mT)g(t\text{-}kMT\text{-}mT\text{-}r_mT)]\ e^{-j\Omega t}\,dt$$
$$= G(\Omega)[\Sigma_{m=0,(M-1)}\Sigma_k\ x(kMT+mT)\ e^{-j\Omega kMT}\ e^{-j\Omega mT}\ exp(-j\Omega\, r_m T)]$$
$$= G(\Omega)\Sigma_{m=0,(M-1)}\ [(1/2\pi)\int X(\omega)\ e^{j(\omega-\Omega)mT}\ \Sigma_k\ e^{jk(\omega-\Omega)MT}\,d\omega$$
$$exp(-j\omega\, r_m T)]$$

Again by applying the identity in Equation *(11)*, we have

$$F_3(\Omega) = G(\Omega)(1/MT)\Sigma_{m=0,(M-1)}\Sigma_k X(\Omega\text{-}k[2\pi/MT])$$
$$exp(-jr_m\Omega T)\ e^{-jkm(2\pi/M)}$$
$$= G(\Omega)(1/T)\Sigma_k A_{3,k}(\Omega)\ X(\Omega\text{-}k[2\pi/MT]) \qquad (16)$$

where

$$A_{3,k}(\Omega) = (1/M)\Sigma_{m=0,(M-1)}\ exp(-jr_m\Omega T)\ e^{-jkm(2\pi/M)} \qquad (17)$$

Equations *(16)* and *(17)* completely characterize the Fourier transform $F_3(\Omega)$. From *(17)* and the Parseval theorem we have

$$\Sigma_{k=0,(M-1)}\ |A_{3,k}(\Omega)|^2 = 1 \qquad (18)$$

If $x(t)$ is a sinusoidal, $exp(j\Omega_o t)$, which has a Fourier transform $X(\Omega) = 2\pi\delta(\Omega\text{-}\Omega_o)$. Then, Equation *(18)* can be used to calculate the signal to noise ratio (SNR) as follows:

$$SNR = 10\ log\ (\ |A_{3,0}(\Omega_o)|^2 / [1\text{-}|A_{3,0}(\Omega_o)|^2]\ ) \qquad (19)$$

where

$$A_{3,0}(\Omega_o) = (1/M)\Sigma_{m=0,(M-1)}\ exp(-jr_m\Omega_o T) \qquad (20)$$

These equations can be used to calculate the signal to noise ratio of a DDS (direct digital synthesizer) output with a jittered master clock [3]. It is also noted that Equations *(19)* and *(20)* are identical to Equations *(15)* and *(16)* except the sign of the exponential functions in *(16)* and *(20)*.

5. Fourier spectrum of D/A outputs with a non-uniformly sampled input signal and a compensating time-varying clock [2]

If the digital data $x[n]$ are obtained from non-uniformly sampling the continuous time signal $x(t)$ and that the D/A clock are adjusted to match the corresponding sampling time instances, then the D/A output is given by $f_4(t) = \Sigma_n x(t_n)g(t\text{-}t_n)$, and we have its Fourier transform as

$$F_4(\Omega) = \int f_4(t)e^{-j\Omega t}\,dt$$

Following the same derivation procedures in the previous two sections, we have

$$F_4(\Omega) = G(\Omega)(1/T)\Sigma_k A_{4,k}\ X(\Omega\text{-}k[2\pi/MT]) \qquad (21)$$

where

$$A_{4,k} = (1/M)\Sigma_{m=0,(M-1)}\ exp(-jr_m k[2\pi/M])\ e^{-jkm(2\pi/M)} \qquad (22)$$

It is noted that $A_{4,k}$ is independent of the frequency Ω, and $A_{4,0} = 1$. It is also noted that $A_{4,k} = A_{2,k}(\Omega)\,|_{\Omega=0}$.
A typical plot of $F_4(\Omega)\ /G(\Omega)$ is shown in Figure 7.

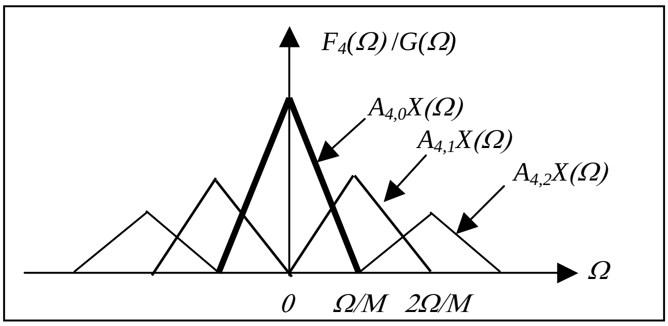

Figure 7

6. Fourier spectrum of D/A output: $f_5(t) = \Sigma_n x(t_n)g_n(t- t_n)$

In this model, the basic output pulse waveforms, $g_n(.)$, are changing from sample to sample. In general, this is a very difficult problem. If we restrict $g_n(.)$ in Equations *(5)* to be a constant-amplitude pulse, then the model can be applied to the situation where digital data with a finite number of bits are fed to an D/A converter with a conventional zeroth-order hold circuit output. We can then investigate if the time varying clock scheme can be used to compensate the quantization error in the input digital data that can be considered as being obtained from non-uniformly sampling a continuous time signal.

A typical plots of the ideal case (i.e., $f_1(t)$)and $f_5(t)$ with zeroth-order hold circuit outputs are shown in Figure 8 below. The difference, *e(t)*, between the two is also shown there.

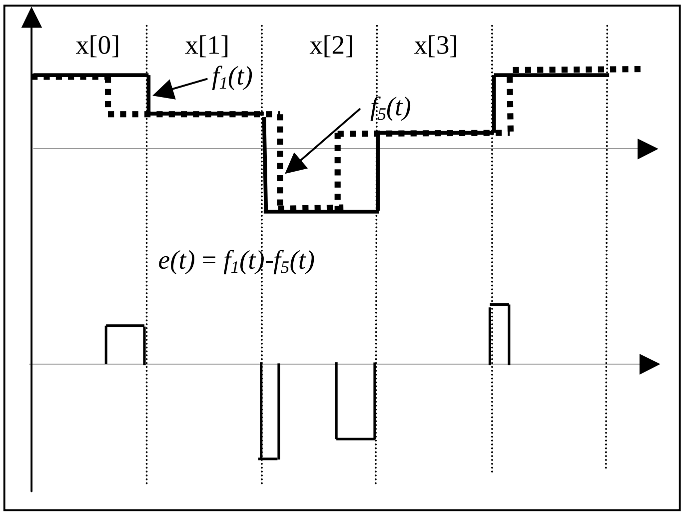

Figure 8

From Figure 8, we can write down the expression of *e(t)* as follows

$$e(t) = \Sigma_n y[n]u_n(t-nT) \qquad (23)$$

where $y[n+1] = x[n+1]-x[n]$ and is $u_n(t)$ is define as follows:

$u_n(t) =$ *1*, for $t \in (0, r_nT)$, if r_nT is positive
0, elsewhere

and

$u_n(t) =$ *-1*, for $t \in (r_nT, 0)$, if r_nT is negative
0, elsewhere

The $u_n(t)$ is shown below in Figure 9.

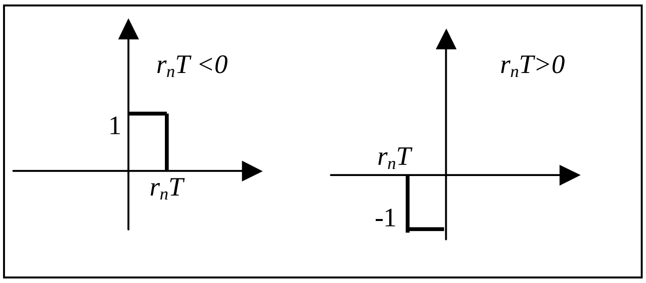

Figure 9

Follow the similar derivation procedure as in previous sections, we can show that the Fourier transform, $E(\Omega)$, of *e(t)* is given by

$$E(\Omega) = (1/T)\, \Sigma_k A_{5,k}(\Omega)\, X'(\Omega-k[2\pi/MT]) \qquad (24)$$

Where

$$A_{5,k}(\Omega) = (1/M)\Sigma_{m=0,(M-1)}\, U_m(\Omega)\, e^{-jkm(2\pi/M)} \qquad (25)$$

and $X'(\Omega)$ is the Fourier transform of $x(t)-x(t-T)$.
It is noted that $U_m(\Omega)$, the Fourier transform of $u_m(t)$, is periodic on *m* with period *M* because it has the same property as r_n. Equations (24) and (25) can be used to compute the Fourier spectrum of *e(t)* and hence that of $f_5(t)$.

7. Conclusions

In this paper, we investigate problems of D/A converters with non-uniformly sampled input data, and/or time-varying clock sources. Recently, there is available a new clocking system with a very fine time resolution and is capable of adjusting the clock period in a sample-to-sample basis. It is, therefore, interesting to consider the following question. "Given that the timing offset of each data sample is known, would it be beneficial to use this offset to adjust the read-out timing of the D/A converter?" To answer this question, we consider the following five different models: $f_1(t) = \Sigma_n x(nT)g(t-nT)$, $f_2(t) = \Sigma_n x(t_n)g(t-nT)$, $f_3(t) = \Sigma_n x(nT)g(t-t_n)$, $f_4(t) = \Sigma_n x(t_n)g(t-t_n)$, and $f_5(t) = \Sigma_n x(t_n)g_n(t- t_n)$. Closed form expressions for the Fourier transform of the output signals for each model are derived. We have also discussed some potential practical applications.

8. References

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