Estimation of Delta–Sigma Converter Spectrum

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Abstract – *Effects of the windowing process, widely investigated by the scientific literature for narrow–band components embedded in white noise, is not sufficiently detailed when signals are corrupted by colored noise. Such a phenomenon can heavily affect the spectral parameters estimation of the noisy signal. In this paper effects of the windowing on the output of analog–to–digital converters with $\Delta\Sigma$ topology, which present a spectrally shaped quantization noise, is analyzed. In particular, the spectral leakage of both narrow– and wide–band components is investigated and a criterion for choosing the most appropriate window for any given modulator resolution is given. The proposed analysis validates the use of the Hanning sequence as the optimum two term cosine window to be employed for characterizing low order $\Delta\Sigma$ modulators.*

Keywords – Delta–Sigma spectrum, spectral leakage, Delta–Sigma characterization.

I. INTRODUCTION

Estimation of spectral figures of merit of analog–to–digital converters (ADCs), such as signal–to–random noise ratio ($SRNR$), signal–to–noise and distortion ratio ($SINAD$), spurious–free dynamic range ($SFDR$) or total harmonic distortion (THD), is usually carried out by employing frequency–domain based techniques. In particular, as described in the standards IEEE 1057 and 1241, such spectral parameters are calculated from the spectrum of the converter output sequence. This is usually estimated using the Discrete Fourier Transform (DFT) based on N acquired samples, that allow the power evaluation of the narrow– an wide–band components [1] [2].

When non–coherent sampling applies, the finite number of processed samples induces spectral leakage phenomena which may affect the estimation of the parameter of interest. In order to reduce such effects, windowing is usually applied to the acquired data [1]–[4]. While this technique has been widely investigated for classical Nyquist–rate ADCs, the analysis of the windowed output spectrum of $\Delta\Sigma$ modulators, which present a spectrally shaped quantization error sequence, is not sufficiently detailed by the published scientific literature.

Nevertheless, such converters are commonly employed in digital measurement and telecommunication systems and the estimation of their spectral parameters is usually carried out in the frequency–domain by weighting the output data with the Hanning sequence.

In this paper, the spectral leakage effects on the $\Delta\Sigma$ modulator output are considered and a criterion for choosing the most appropriate window for a given modulator resolution is given. Such analysis validates the use of the Hanning sequence as the most suitable window to be employed for accurately estimating parameters of both narrow– and wide–band components. Moreover, it is analyzed the effect of the windowing process on the power spectrum of the modulator quantization error.

II. $\Delta\Sigma$ OUTPUT SPECTRUM

The architecture of a generic $\Delta\Sigma$ modulator is composed by an L–th order loop filter, a quantizer and a negative feedback loop. In spite of its apparent simple structure and because of the presence of the quantizer, the analysis of its performance is rather complex. In order to simplify this task, the internal ADC is often modeled as an additive error signal $e[\cdot]$ uniformly distributed and uncorrelated with the input signal $x[\cdot]$. By assuming such a linear model, the output of an L–th order modulator can be written as:

$$y_L[n] = x[n-L] + q_L[n] \qquad (1)$$

where $q_L[\cdot]$ is the quantization error of the L–th order shaped modulator.

Estimates of $SRNR$, $SINAD$ or THD can be carried out by stimulating the modulator with a sinusoidal signal and by applying the FFT algorithm to R data records each of length N. The finite length of the processed signal and the discretization of the frequency axis, induce an estimation bias on the ADC output spectrum.

In order to reduce the spectral leakage phenomenon, the modulator output samples are usually multiplied by an appropriate window, $w[\cdot]$. Sequences commonly employed are those belonging to the cosine–class because they can easily be calculated. In this paper, $w[\cdot]$ has been normalized to the square root of the energy value in order to bound the maximum of the correlation sequence values to 1. By indicating with $\overline{w}[\cdot]$ the normalized window,

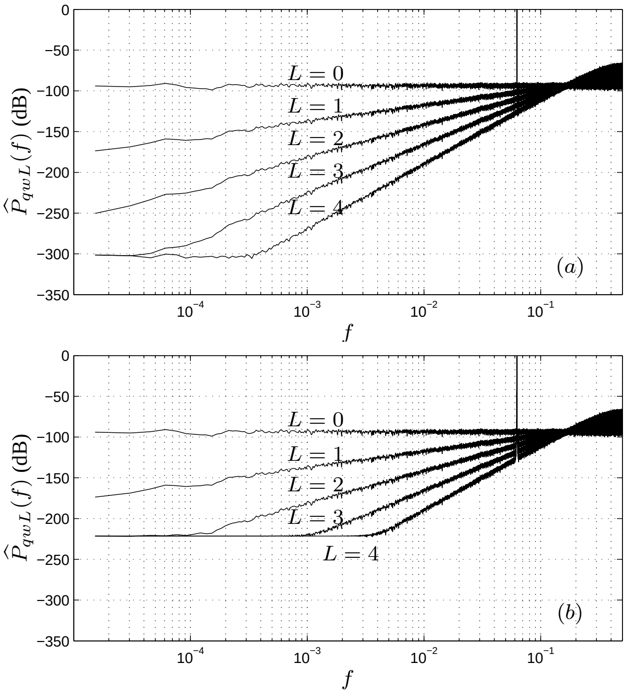

Figure 1. Power spectral densities of the Hanning–windowed output of an 8–bit modulator stimulated by a sinewave with amplitude equal to the internal ADC full–scale under coherent (a) and non–coherent (b) sampling. Lines in each figure refer to five different order-shaping L as indicated by the corresponding labels. The input frequency value has been set to the modulator upper–band edge by considering an $OSR = 8$.

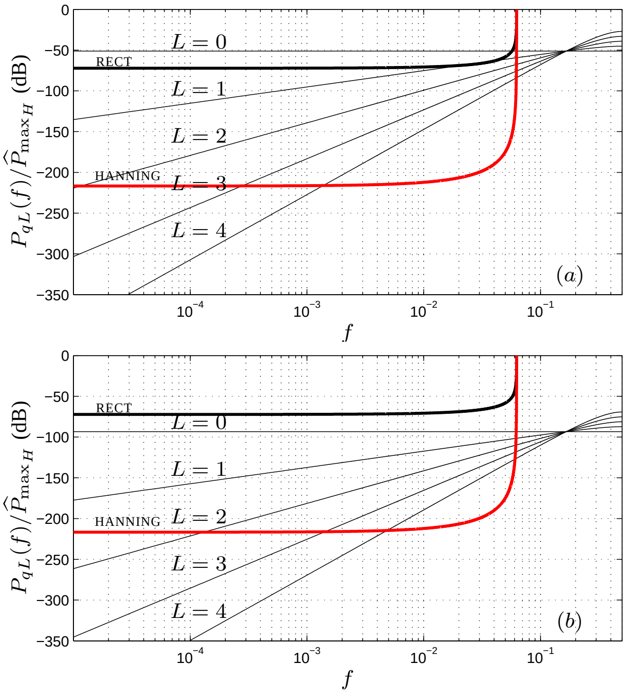

Figure 2. Solid lines represent the power spectral densities of the 1–bit (a) and 8–bit (b) L–order–shaped quantization noise, normalized to $\widehat{P}_{\max_H}$. Bolded lines represent the sidelobe envelope of the windows indicated by the corresponding labels, and centered on the modulator upper–band edge when considering an $OSR = 8$.

it follows that the analyzed signal, $y_{L\overline{w}}[n]$, is:

$$y_{L\overline{w}}[n] \stackrel{\triangle}{=} y_L[n]\overline{w}[n] = (x[n-L] + q_L[n])\overline{w}[n], \quad n = 0, ..., N-1. \tag{2}$$

Simulation results presented in this paper, refer to a modulator output weighted by the Hanning window. This is the sequence most commonly employed for estimating spectral parameters of $\Delta\Sigma$ modulators, because of its suitable properties in the frequency domain.

In particular, such a window presents a high sidelobe envelope decay together with a small mainlobe width (i.e. 4 frequency bins) thus optimizing the requirements of frequency selectivity and low spectral leakage. As a consequence, a high resolution quantizer is needed in order to evaluate effects of spectral leakage associated with the narrow–band component.

Fig. 1 shows the power spectral densities, estimated by means of the periodogram, $\widehat{P}_{qL\overline{w}}(f)$, of an 8–bit $\Delta\Sigma$ converter with an oversampling ratio (OSR) equal to 8, when stimulated by a full–scale (FS) input sinewave, for different modulator order–shaping, as indicated by the corresponding labels. The frequency axis has been normalized to the converter sampling rate. The input frequency value has been set equal to $f_0 = 1/(2 \cdot OSR)$, i.e. on the modulator upper–band edge. Moreover, the modulator output has been windowed by the Hanning sequence, and the periodogram has been carried out on $R = 50$ non–overlapped data records each of length $N = 2^{16}$. Fig. 1(a) and (b) refer to coherent and non–coherent sampling of the input sinewave frequency, respectively, both normalized to the maximum of the periodogram under the coherent condition, i.e. $\widehat{P}_{\max_H} \stackrel{\triangle}{=} FS^2/(4 \cdot ENBW)$, where $ENBW \stackrel{\triangle}{=} N\sum_{n=0}^{N-1}\overline{w}^2[n]/\left(\sum_{n=0}^{N-1}\overline{w}[n]\right)^2$ is the equivalent noise bandwidth of the window.

In both figures, the power spectral densities of high–order modulators converge to a constant value in the low–frequency band. When non–coherent sampling applies, such a phenomenon is mainly due to the spectral leakage of the narrow–band component. Thus, in the low–frequency region of Fig. 1(b), samples of the window sidelobes are graphed. However, when coherency is guaranteed, since zeros of the window spectrum are located on integer frequency bin values, such a phenomenon is not due to the narrow–band component, but only to spectral leaking of the shaped–quantization noise.

Because of the discrete resolution of the frequency axis, which is equal to $1/N$, the periodogram shows such a phenomenon only by employing a high number of samples N.

In the following, the effects of the spectral leakage of the narrow–band component and of the shaped–quantization noise are analyzed and discussed separately.

III. Spectral leakage of the narrow–band component

The power spectral density of the shaped noise $q_L[\cdot]$ can easily be calculated by applying the Discrete Time Fourier Transform (DTFT) to its autocorrelation sequence, thus obtaining [6]:

$$P_{q_L}(f) = \sigma_e^2 2^{2L} \sin^{2L}(\pi f), \quad |f| < 0.5, \qquad (3)$$

where σ_e^2 is the quantization error power of the internal ADC and f is the normalized frequency. The behavior of (3), normalized to $\widehat{P}_{\max_H}$, is shown in Fig. 2(a) and (b) (solid lines) for a 1–bit and an 8–bit modulator, respectively, for different orders L, as indicated by the corresponding labels.

To determine the effects of spectral leakage of the narrow–band component on the modulator output spectrum, the envelope of the sidelobes of the rectangular and Hanning windows have been also plotted in Fig. 2(a) and (b) with bolded lines for $N = 2^{16}$. The envelopes have been traced only for frequencies lower than that of the input tone since effects of the window sidelobes at higher frequencies can be neglected because of the higher power of the shaped noise. Moreover, the image component of the narrow–band signal can be neglected, thus providing a simpler analysis.

As a consequence, the behavior of the rectangular and Hanning windows can be approximately described by $1/|f_0 N - fN|$ and $1/|f_0 N - fN|^3$, respectively [5].

The spectral leakage of the narrow–band component can affect the modulator output spectrum only if the shaped–noise power spectral density is lower than the envelope of the used window, especially when high OSRs are employed. As an example, Fig. 2(b) shows that, by applying the Hanning window, spectral leakage effects appear only for modulator orders higher than or equal to 2.

For low frequency values, the sidelobe envelopes of the rectangular and Hanning windows converge, respectively, to $1/(f_0 N)$ and to $1/(f_0 N)^3$. By setting these values equal to (3), the frequency f_x over which the leakage of the narrow–band component do no affect the spectral estimation, can be easily determined. In particular, since for small f the condition $\sin^{2L}(\pi f_x) \simeq (\pi f_x)^{2L}$ holds true, for any given L–th order shaping modulator, windowed by a rectangular or a Hanning window of length N, such frequency values can be expressed, respectively, as:

$$f_{xR} = K \cdot W, \quad f_{xH} = K^3 \cdot W \qquad (4)$$

where $K = (1/f_0 N)^{1/2L}$ and $W = 1/(2\pi\sigma_e^{1/L})$.

Thus, the spectral leakage of the narrow–band component can be made arbitrary small by decreasing the quantizer resolution (i.e. reducing σ_e) or the modulator shaping order or by increasing the number of processed samples N.

Notice that, since the first and the last samples of the Hanning window are equal to 0, the Hanning sequence is the two term window which presents a sidelobe decay equal to $1/f^3$ instead of $1/f$ [5]. Thus, it is the most suitable two term window to employ for reducing spectral leakage effects on the modulator output spectrum, at least for order shaping lower than 2. Higher order modulators require windows with more coefficients.

IV. Effect of windowing on the shaped–quantization noise

The power spectral densities of the $\Delta\Sigma$ shaped noise, windowed by the rectangular and the Hanning sequence, have been calculated as indicated in App. A. In particular, the expressions of the power spectral densities of the noise filtered by a first and second order modulator, and windowed by a rectangular sequence, result to be respectively equal to:

$$P_{q_1\overline{w}_R}(f) = P_{q1}(f) + 2\sigma_e^2 \cos(2\pi f)/N, |f| < 0.5 \qquad (5)$$

$$P_{q_2\overline{w}_R}(f) = P_{q2}(f) + 2\sigma_e^2(4\cos(2\pi f) + -2\cos(4\pi f))/N, \quad |f| < 0.5. \qquad (6)$$

Expressions (5) and (6) show that the estimated power spectral densities of the first and second order shaped noise converge, in the low frequency region, to a constant value approximately equal to $2\sigma_e^2/N$ and $4\sigma_e^2/N$, respectively.

By applying the same procedure, the expressions of power spectral densities of the filtered noises, windowed by the Hanning sequence, are:

$$P_{q_1\overline{w}_H}(f) = P_{q_1}(f) + 2\sigma_e^2\left(1 + -R_{\overline{w}_H}[1]\cos(2\pi f)\right), \quad |f| < 0.5 \qquad (7)$$

$$P_{q_2\overline{w}_H}(f) = P_{q_2}(f) + 2\sigma_e^2\left(R_{\overline{w}_H}[2]\cos(4\pi f) + -4R_{\overline{w}_H}[1]\cos(2\pi f) + 3\right), \quad |f| < 0.5 \qquad (8)$$

where $R_{\overline{w}_H}[1]$ and $R_{\overline{w}_H}[2]$ are constant values that can be calculated from (A.6).

As in the previous case, (7) and (8) show that for low frequency values, the power spectral densities of the noise shaped by a first and second order modulator, converge, respectively, to constant values approximately equal to

$$K_{H1} = 2\sigma_e^2 \left(1 - R_{\overline{w}_H}[1]\right) \quad (9)$$

$$K_{H2} = 2\sigma_e^2 \left(R_{\overline{w}_H}[2] - 4R_{\overline{w}_H}[1] + 3\right) \quad (10)$$

which can be made arbitrarily small by increasing the number of acquired samples N.

By substituting in (9) and (10), the values of σ_e^2 and N employed in Fig. 1, we have $K_{H1} = -148.5$ dB and $K_{H2} = -249.5$ dB.

Thus, the leakage of the wide–band components, due to the truncation of the modulator output sequence, affects the estimation of the power of the modulator output error when a low N is employed, especially when high OSRs are considered.

V. Conclusions

In this paper the windowing process on $\Delta\Sigma$ modulator output has been analyzed. The effect of the spectral leakage of the narrow–band component and of the spectrally shaped–quantization noise on the estimated power spectral density has been discussed. In particular, it has been shown that the spectral leakage of the narrow– and wide–band component can be made arbitrarily small by increasing the number of processed samples and by properly choosing a window sequence.

Windows most commonly employed are those attaining to the two term cosine class since they present a small mainlobe width and are easy to calculate. The presented analysis has shown that the Hanning sequence is the two term window which guarantees the lower estimation bias of spectral parameters, thus validating its use in the characterization of $\Delta\Sigma$ modulators.

Appendix A

Derivation of expressions (5)–(8)

By indicating with $\mathcal{F}\{\cdot\}$ the Fourier Transform operator, the power spectral density of a windowed L–th order shaped noise $q_{L\overline{w}}[n] \stackrel{\triangle}{=} q_L[n]\overline{w}[n]$ with autocorrelation function $R_{q_{L\overline{w}}}[m]$ is:

$$P_{q_L\overline{w}}(f) = \mathcal{F}\{R_{q_L\overline{w}}[m]\} = \mathcal{F}\{R_{q_L}[m]R_{\overline{w}}[m]\} \quad \text{(A.1)}$$

where $R_{q_L}[m]$ represent the autocorrelation function of the L–th order shaped noise and

$$R_{\overline{w}}[m] = \sum_{n=0}^{N-1} \overline{w}[n+m]\overline{w}[n], \quad \text{(A.2)}$$
$$m = -(N-1), ..., (N-1)$$

is the aperiodic correlation sequence of the employed normalized window.

The autocorrelation sequence of a white noise with zero–mean and variance equal to σ_e^2, filtered by a first and second order modulator, respectively, result to be [6]:

$$R_{q_1}[m] = \sigma_e^2 \left(2\delta[m] - \delta[m-1] - \delta[m+1]\right) \quad \text{(A.3)}$$

$$R_{q_2}[m] = \sigma_e^2 \left(6\delta[m] - 4\delta[m-1] - 4\delta[m+1] + \right.$$
$$\left. + \delta[m-2] + \delta[m+2]\right) \quad \text{(A.4)}$$

where $\delta[\cdot]$ is the discrete Dirac pulse.

By applying (A.2), the autocorrelation of the normalized rectangular and Hanning windows result to be, respectively:

$$R_{\overline{w}_R}[m] = 1 - \frac{|m|}{N}, \quad m = -(N-1), ..., (N-1) \quad \text{(A.5)}$$

$$R_{\overline{w}_H}[m] = \frac{2}{3} - \frac{2}{3N}|m| + \frac{1}{3N}\cos\left(\frac{2\pi}{N}|m|\right)(N - |m|) +$$
$$+ \frac{2}{3N}\left(\frac{\sin\left(\frac{2\pi}{N}\right)}{1-\cos\left(\frac{2\pi}{N}\right)} - \frac{\cos\left(\frac{2\pi}{N}\right)}{2\sin\left(\frac{2\pi}{N}\right)}\right)\sin\left(\frac{2\pi|m|}{N}\right),$$
$$m = -(N-1), ..., (N-1) \quad \text{(A.6)}$$

By substituting (A.3) and (A.4) in (A.1) and considering that for real windows $R_{\overline{w}}[m] = R_{\overline{w}}[-m]$, expressions (5)–(8) result.

References

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