UNCERTAINTY ESTIMATION BASED ON INVERSE MODELS: AN OPEN QUESTION

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Abstract – *In measurement science and its technological application most of the measurement methods are indirect. In order to measure the unknown physical quantity y we have to develop a forward model which relates this quantity to another quantity x directly measurable: $x \rightarrow y$. Often the measurement model available is of opposite nature, i.e. $y \rightarrow x$. It is thus necessary to invert the available model; this operation in some cases can lead to unacceptable level of uncertainty on the results.*
This paper shows the properties of some algorithms for processing measured data using ill posed inverse models employed for determining the distribution of indirectly measured quantities.
The inversion procedure requires regularisation techniques in order to limit the uncertainty affecting the indirect measurements.

Keywords - Uncertainty analysis, process tomography, signal processing, least squares methods.

1. INTRODUCTION

It is well known how in measurement science and its technological application many, indeed most, measurement methods are indirect. In order to measure the unknown physical quantity y we have to develop a model K[·] which relates this quantity to another quantity x directly measurable:

$$y = \mathrm{K}[x] \tag{1}$$

where K[·] is an appropriate operator (the model). It will be assumed in the following that the direct measurements x are represented by a real vector or matrix with M data. The indirect measurement y will be represented instead either as a real vector or a real matrix with N data or as a real function ($N=\infty$). This latter analytical form is useful for describing the space distribution of physical quantities. Anyway it will be always assumed that $N>M$. Models of type (1) are known as *forward* models.

In practice what is often available is a model of the form:

$$x = \mathrm{A}[y] \tag{2}$$

where A is the appropriate operator.

In this case the indirect measurement can be accomplished inverting the operator A[·]:

$$y = \mathrm{A}^{-1}[x] \tag{3}$$

Expression (3) represents the *inverse* model [1,2].

This apparently trivial mathematical operation is not always so simple from a metrological and mathematical point of view. As a matter of fact many inverse models $\mathrm{A}^{-1}[\cdot]$ of practical interest belong to the so called class of *ill posed problems*, where a small amount of uncertainty in the direct measurements x introduce an unlimited uncertainty in the indirect measurements y through (3).

In expression (2) and (3) the directly and indirectly measured quantities x and y can be the response of the monitored or tested system to a known input exciting quantity u:

$$x(u) = \mathrm{A}[y,u] \tag{4}$$

$$y(u) = \mathrm{A}^{-1}[x,u] \tag{5}$$

The results of the diagnostic experiment described by the inverse model (5) are usually known as *process tomography* [3]. A great variety of different physical quantities can be used for excite the response of the system under observation (acoustic waves, electromagnetic waves, electrical currents, ionising radiation, etc.) giving rice to different kind of process tomography.

Typically indirect measurements x are obtained adopting the weighted least squares techniques, minimising the "distance" d between the direct measurements y and their values predicted by the forward models (2) and (4):

$$d(x)^2 = \left\|\mathbf{A}[x] - y\right\|_{C_y}^2 \tag{6}$$

where C_y in the 2-norm (6) is the variance-covariance matrix of the direct measurements uncertainties.

When $\mathrm{A}^{-1}[\cdot]$ is ill-posed the lest squares method leads to an unacceptable uncertainty on x and a constrained least squares approach is mandatory. The constraints are obtained from some a-priori knowledge on some functional characteristics of the distribution of the indirectly measured quantity x.

The above approach to indirect measurements, although typical in remote sensing and non-destructive testing, raises important questions about the criteria for estimating the uncertainty of the indirect measurement results. As a matter of fact most of the research results on this subject deal with the methodologies for obtaining the indirect measurements, but very little has been done for estimating their uncertainties.

This paper approach the problem of the uncertainty analysis of indirect measurements obtained adopting ill-posed inverse models. After a short summary of the Tikhonov regularisation techniques for the inversion of ill-posed models, two examples of uncertainty analysis in the field of electrical measurements will be developed: the reconstruction of current distribution in a conductor and the electrical impedance tomography.

2. A LINEAR MODEL: THE MEASUREMENT OF CURRENT DENSITY DISTRIBUTIUON

An example of inverse problem in electrical measurements is represented by the relationship between the current distribution $\vec{J}$ in a conductor and the magnetic field $\vec{B}$ as represented by the Law of Biot and Savart:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \iiint_{V'} \frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{\left\| \vec{x} - \vec{x}' \right\|^3} dV' = \mathrm{A}\left[\vec{J}\right] \qquad (7)$$

In (7) $\vec{x}$ is the vector pointing to the position where the magnetic field induction is measured and $\vec{x}'$ is the vector pointing to a generic position inside the volume *V'* where the current is flowing.

The inverse operator $\mathrm{A}^{-1}[\cdot]$ can be used for reconstruction of the current density distribution $\vec{J}$ in the volume *V'* from direct measurements of the magnetic field around it. When (7) is used as an inverse model for indirectly measure the density current distribution from the direct measurement of the magnetic field we incur in an ill-posed inverse problem. Its inversion is critical and requires special techniques for obtaining a "stable" solution.

For the sake of simplicity in the following we will consider the 1D problem made up by thin planar current layer distributed in an un-limited conductor of width *L*, as shown in figure 1. Only the magnetic filed density vector component parallel to the conductor plane is considered, at a distance *d* from the conductor.

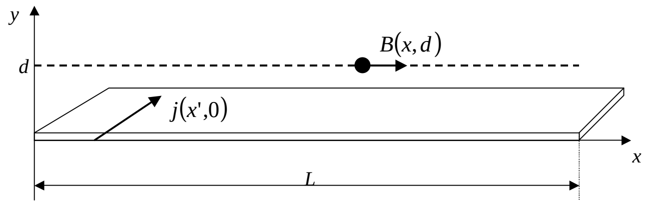

Fig.1 – 1D electromagnetic problem relating the current density vector to the x-component of the magnetic flux density at distance *d*.

In order to study the metrological properties of the operator $\mathrm{A}[\cdot]$ in (7) let us start with the forward problem, i.e. the indirect measurement of the *x*-component of the magnetic flux density at the generic position of coordinate (*x*,*d*) from the direct measurement of the current density at *N* positions with coordinates (x'_i,0). Obviously this hypothetical experimental situation is theoretical; its interest is associated only to the comprehension of the metrological behaviour of the operator $\mathrm{A}[\cdot]$.

The continuous model (7) can be approximated by a finite difference representation considering the magnetic field induction in *M* positions and the current density constant around the *N* measurement positions. The operator $\mathrm{A}[\cdot]$ is thus transformed in a $M \times N$ matrix $\mathbf{A}_\mathrm{D}$ and the continuous model (7) in the following matrix relationship:

$$\mathbf{B} = \mathbf{A}_\mathbf{D} \cdot \mathbf{j} \qquad (8)$$

where **B** is the $M \times 1$ vector of the indirect measurements of the magnetic field density and **j** is the $N \times 1$ vector of the direct measurements of the density current.

Figure 2 shows the results obtained with *N*=5 and *M*=3,5 and 10 respectively (the measurements positions are equally spaced). Also the error bars are displayed showing the expanded uncertainty [4,5] computed considering a coverage factor *k*=2. The expanded uncertainty has been estimated considering the measurement model (8) under the hypothesis of a 5% relative standard uncertainty in the measurement of the current density. Figure 2 shows also the theoretical magnetic field density computed using (7).

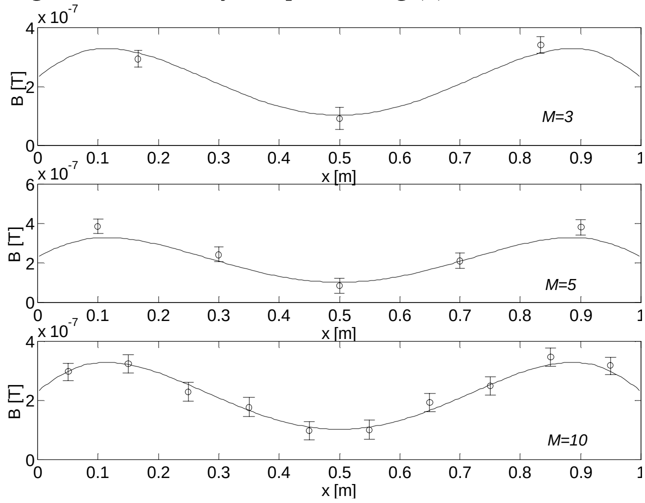

Fig.2 – x-component of the magnetic flux density at distance *d* at *M*=3,5 and 10 positions respectively from the measurement of the current density at *N*=5 positions. The continuous line shows the actual value of the magnetic field.

Let us consider now the more interesting problem of indirectly measure the density current distribution in *N* positions from *M* measurements of the magnetic flux density. Considering the approximated forward model (8) one possibility of inversion consists in finding the least square solution by the normal equation [6]:

$$\tilde{\mathbf{j}} = \left[\left(\mathbf{A}_\mathbf{D}^\mathbf{T} \cdot \mathbf{A}_\mathbf{D}\right)^{-1} \cdot \mathbf{A}_\mathbf{D}^\mathbf{T}\right] \cdot \mathbf{B} = \mathbf{\Gamma} \cdot \mathbf{B} \qquad (9)$$

In (9) the $N \times 1$ vector $\tilde{\mathbf{j}}$ is the solution minimizing the distance functional *d* as defined in (6) and applied to (8):

$$d(\mathbf{j})^2 = \left\| \mathbf{A}_\mathbf{D} \cdot \mathbf{j} - \mathbf{B} \right\|^2 = \left[\mathbf{A}_\mathbf{D} \cdot \mathbf{j} - \mathbf{B}\right]^T \cdot \left[\mathbf{A}_\mathbf{D} \cdot \mathbf{j} - \mathbf{B}\right] \qquad (10)$$

The matrix $\mathbf{\Gamma}$ in (9) appears as the inverse model for the proposed measurement problem. Figure 3 shows the results obtained applying (9) with *M*=5 and *N*=3,5 and 10 respectively (the measurements positions are equally spaced). Also the error bars are displayed showing the expanded uncertainty computed considering a coverage factor *k*=2. The expanded uncertainty has been estimated considering the measurement model (9) under the hypothesis of a 5% relative standard uncertainty in the measurement of the magnetic flux density.

Figure 3 shows also the theoretical current density used for computing the magnetic flux density using the continuous model (7). It appears that because of the ill posed nature of the operator $A^{-1}[\cdot]$ its discrete approximation $\mathbf{\Gamma}$ is strongly ill conditioned.

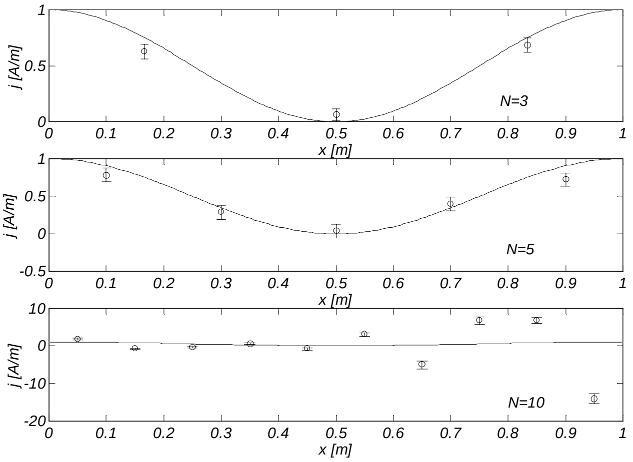

Fig.3 - Current density at N=3,5 and 10 positions respectively from the direct measurement of the x-component of the magnetic flux density at M=5 positions. The continuous line shows the actual value of the current density.

In order to gain a better understanding of the problem let us split relation (9) as follows:

$$\mathbf{y} = \mathbf{A}_\mathbf{D}^\mathbf{T} \cdot \mathbf{B} \tag{11}$$

$$\tilde{\mathbf{j}} = \left(\mathbf{A}_\mathbf{D}^\mathbf{T} \cdot \mathbf{A}_\mathbf{D}\right)^{-1} \cdot \mathbf{y} \tag{12}$$

Figure 4 shows **y** and $\tilde{\mathbf{j}}$ for N=3,5 and 10 respectively.

From this figure it is possible to see that while the expanded uncertainty on **y** is limited the expanded uncertainty on $\tilde{\mathbf{j}}$ reach unacceptable values for N=10. This behavior is connected to the structure of the matrix $\left(\mathbf{A}_\mathbf{D}^\mathbf{T} \cdot \mathbf{A}_\mathbf{D}\right)^{-1}$ in (12) whenever $N>M$.

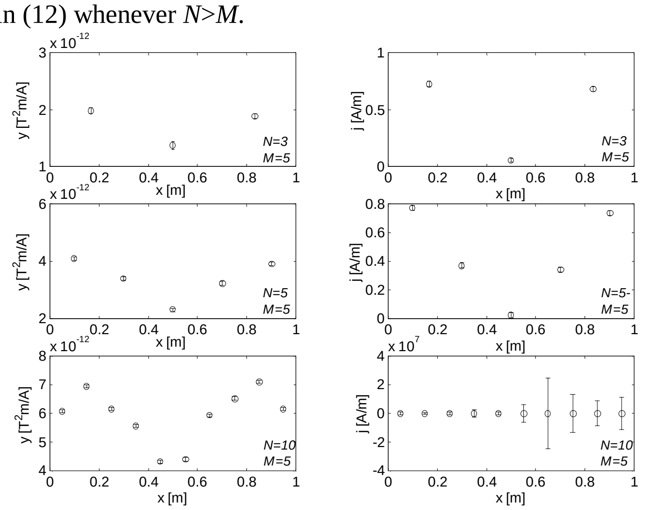

Fig.4 – Effects of the split inversion operator (11) and (12) on the reconstruction of the current density distribution.

In order to avoid this drawback the inversion of the forward model (8) requires a different approach than the least squares method. If we reconsider the distance functional (6) an alternative definition is as follows: [2,7]:

$$d(\mathbf{j})^2 = \left\|\mathbf{A}_\mathbf{D} \cdot \mathbf{j} - \mathbf{B}\right\|^2 + \alpha\left\|\mathbf{j}\right\|^2 \tag{13}$$

In (13) there are two terms. The first measures the disagreement with observation; the second measures the size of the solution, i.e. its non-smoothness.

Minimisation of (13) lead to the following solution:

$$\tilde{\mathbf{j}} = \left[\left(\mathbf{A}_\mathbf{D}^\mathbf{T} \cdot \mathbf{A}_\mathbf{D} + \alpha\mathbf{I}\right)^{-1} \cdot \mathbf{A}_\mathbf{D}^\mathbf{T}\right] \cdot \mathbf{B} = \mathbf{\Gamma}_\mathbf{T} \cdot \mathbf{B} \tag{14}$$

The solution obtained with (13) and (14) is known as Tikhonov regularisation. Figure 5 shows the current density obtained with N=50 and M=5 adopting the regularised solution (14). Also the error bars are displayed showing the expanded uncertainty computed considering a coverage factor k=2. The expanded uncertainty has been estimated considering the measurement model (14) under the hypothesis of a 5% relative standard uncertainty in the measurement of the magnetic flux density.

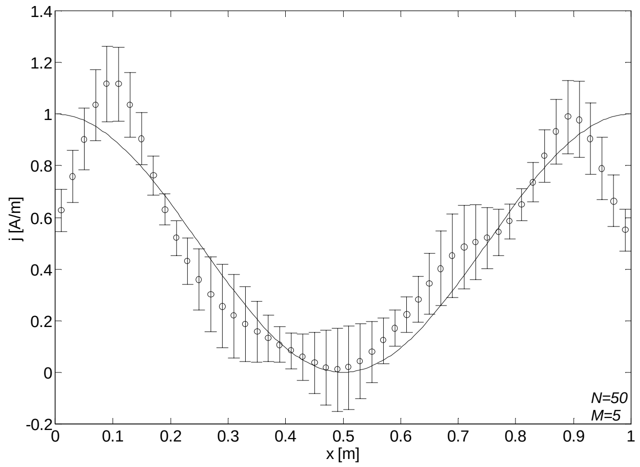

Fig.5 - Current density at N=50 positions obtained from the direct measurement of the x-component of the magnetic flux density at M=5 positions (Tikhonov regularisation). The continuous line shows the actual value of the current density.

3. A NON-LINEAR MODEL: THE MEASUREMENT OF RESISTIVITY DISTRIBUTIUON

An example of process tomography in electrical measurements is the electrical impedance tomography (EIT). It is an attractive example of inverse problem as a technique for non-destructive diagnosis of conductive materials. It has been successfully employed to a wide range of problems including chemical process engineering, biomedical, geophysics and environmental applications.

I would like to restrict our attention specifically to *resistive* EIT in which the objective is to determine the distribution of resistivity ρ within the body of interest from a series of M boundary measurements of potential differences V under M different current injection configurations.

The non-linear relationship between the resistivity distribution ρ and the potential distribution ϕ is represented by the elliptical partial differential equation:

$$\nabla \cdot \left(\frac{1}{\rho}\nabla\phi\right) = 0 \tag{15}$$

subject to the boundary condition imposed by the current injection configurations. The solution of equation (15) is equivalent to a non-linear operator (the forward model) of the form:

$$V = \phi_2 - \phi_1 = F[\rho] \qquad (16)$$

By means of numerical solvers (finite difference, finite elements, boundary elements, etc.) the operator F[·] can be approximated by a discrete non-linear operator $F_D[\cdot]$:

$$\mathbf{V} = F_D[\boldsymbol{\rho}] \qquad (17)$$

where $\boldsymbol{\rho}$ is the $N\times 1$ vector of the resistivity in the N conductor elements.

Again for the sake of simplicity let us consider the 1D problem depicted in figure 6 where the resistivity distribution $\rho(x)$ is unknown.

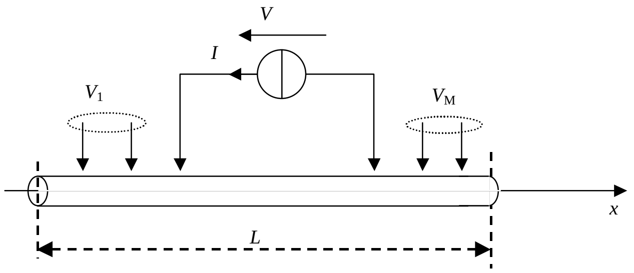

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fig.6 - 1D resistive electrical impedance tomography

The inversion of the forward model (15)-(16) can be obtained by minimization of distance regularizing functional d of this kind [1,7]:

$$d(\boldsymbol{\rho})^2 = \left\|\mathbf{V} - F_D[\boldsymbol{\rho}]\right\|^2 + \alpha\left\|\boldsymbol{\rho} - \boldsymbol{\rho}_b\right\|^2 \qquad (18)$$

where **V** is the vector of M boundary measurements, $\boldsymbol{\rho}$ is the vector of N parameters (here the elements resistivity in the numerical approximation (17)), $F_D[\cdot]$ is the numerical direct model, $\boldsymbol{\rho}_b$ is the vector of background conductivity and $\alpha>0$ is a regularization factor.

Figure 7 shows the least squares and regularized solutions with M=5 current injections and with resistivity evaluated on M=500 positions. Figure 2 shows also the real resistivity distribution.

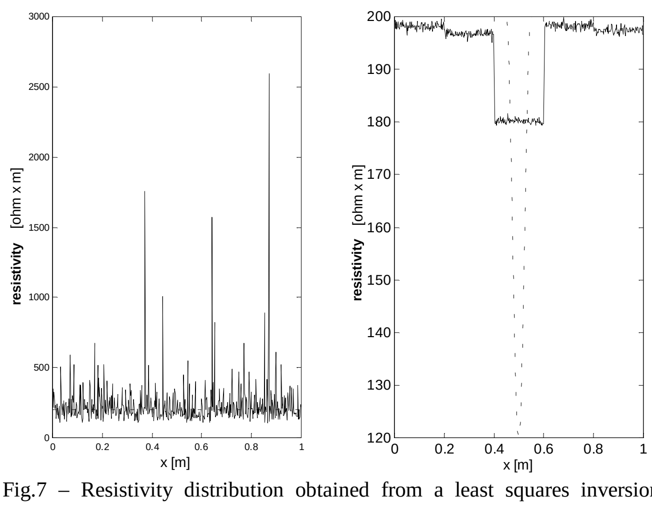

Fig.7 – Resistivity distribution obtained from a least squares inversion method (left) and by Tikhonov regularisation (right). The dotted line shows the actual value of the resistivity.

4. CONCLUSION

This paper introduces the properties of some algorithms for processing measured data using ill posed inverse models employed for determining the distribution of indirectly measured quantities.

The inversion procedure requires regularisation techniques in order to limit the uncertainty affecting the indirect measurements.

REFERENCES

[1] C.W.Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg, Braunschweig/Wiesbaden, 1993.
[2] A.N.Tikhonov, V.Y.Arsenin, *Solutions of Ill-posed Problems*, Winston and Sons, Washington, 1977.
[3] F.Natterer, *The Mathematics of Computerized Tomography*, Teubner, Stutgart, 1986.
[4] S.Brandt, *Statistical and Computational Methods in Data Analysis*, Noth-Holland, Amsterdam, 1976.
[5] ISO, IEC, BIPM, OIML, *Guide to the expression of uncertainty in measurement*, 1992.
[6] D.L.Phillips, *A technique for the numerical solution of certain integral equations of the first kind*, Journal of the Association for Computing Machinery, Vol.9, pagg. 84-97, 1962.
[7] R.L.Parker, *Geophysical Inverse Theory*, Princeton University Press, Princeton, 1994.