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UNCERTAINTY CONTRIBUTION IN THE CASE OF COSINE FUNCTION WITH ZERO ESTIMATE – A PROPOSAL

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Abstract − The model function of the torque generated in a static torque standard machine contains a cosine function for the inclination of the lever against the horizontal plane. An inclined lever has a shorter effective lever arm length, and a smaller torque is generated in this case. The uncertainty analysis of this model function gives a zero sensitivity coefficient for the inclination angle when the estimate of the angle is zero – a mathematical correct, but from the uncertainty point of view, obviously improper result. A method for dealing with this problem is proposed in this paper.

Keywords: uncertainty, cosine function, zero estimate

1. INTRODUCTION

The torque generated in a torque standard machine with a supported lever-mass-system [1] can be given by:

$$M = m\,g\,l\left(1-\frac{\rho_{\mathrm{air}}}{\rho_{\mathrm{m}}}\right)\cos(\alpha)+\sum_{i=1}^{m}M_i+\sum_{j=1}^{n}l\,F_j\,\cos(\alpha) \qquad (1)$$

with the following designations:

m – the mass of the dead-weights in use
g – the local gravitational acceleration
l – the length of the lever arm
ρ_{air} – the density of the air
ρ_{m} – the density of the material of the weights
α – the lever's inclination angle against the horizontal.

There are different effects causing m additional torques M_i and n additional forces F_j; the most important ones are:

M_{P} – pendulum torque of the unbalanced lever
M_{R} – frictional torque of the bearing
M_{D} – driving torque of the bearing
F_{air} – forces due to air flow between the weights
F_{mag} – magnetic forces between the weights.

This equation can be taken as the mathematical model for the determination of the associated measurement uncertainty of the realized torque values. For this purpose, it is necessary to know the uncertainty contribution of the quantities defining the torque. The GUM [2] procedure is applied here, requiring that the derivatives of the model function – with respect to all of the input quantities – are defined.

2. THE PROBLEM

It is not necessary to calculate all these derivatives to see the drawback of the model function in this special case. It is the cosine function of the inclination angle, the derivative of which is a sine function. When the estimate of the input quantity is 0 rad, the value of the derivative and, hence, the sensitivity coefficient, is zero.

From the mathematical point of view, this result is correct. This also becomes obvious when the cosine function is written as a Taylor series near zero:

$$\cos(\alpha)\approx 1-\frac{\alpha^2}{2}+\frac{\alpha^4}{24}-\ldots \qquad (2)$$

Again, the value of the derivative of this series is zero for $\alpha = 0$ rad. Or one can say that the torque change ΔM behaves – near zero inclination – like the square of the inclination angle α, when the terms of higher order are ignored. In this case we have:

$$\Delta M \sim \alpha^2 \ll \alpha \ \text{ for } \ \alpha \ll 1 . \qquad (3)$$

From the uncertainty point of view, a zero sensitivity coefficient would mean that there is no contribution of this quantity to the uncertainty budget of the quantity to be determined, which in this case is the torque. But the inclination angle deviates from zero and when it is measured an uncertainty must be associated with the measurement result and it will surely contribute to the overall uncertainty.

Another problem arises when the value of the inclination angle is not measured each time a calibration is carried out. For calculating the uncertainty of the torques generated in a torque calibration facility, it is often sufficient to take reference values for some of the quantities. In fact, only two of the six basic input quantities of (1) can be considered to be constant: the mass m and the material density ρ_{m} of the weights. The other four quantities change: the local gravitational acceleration g changes due to tidal effects, the length of the lever arm l due to temperature variations and loading, the density of the ambient air ρ_{air} due to temperature and air pressure variations (and some other effects like the composition of the air, for example, its water content), and the inclination angle α due to the elastic deformation of the set-up and the transducer under test, which is often compensated by a counter drive – in this case the residual inclination must be used for the uncertainty estimation.

It is not necessary to measure the values of these four quantities every time, but one can take their reference values, which could be

- the mean of the gravitational acceleration g over a day,
- the measured length l of the unloaded lever,
- the density ρ_{air} of the air under standard conditions and
- the target value of the inclination angle $\alpha = 0$ rad.

Examples for those values are given in Table 1.

Table 1. Reference values and their standard uncertainties for some input quantities (examples for $M = 1000$ N·m).

Quantity	Reference value	Standard uncertainty
g	9.812524 m/s^2	5 µm/s^2
l	0.500000 m	3.5 µm
ρ_{air}	1.20 kg/m^3	0.0462 kg/m^3
α	0.000 rad	1.25 mrad

The maximum deviations could then be defined as the maximum of the absolute differences between the minimum or maximum values that can be observed and the above defined reference values. Especially the density of the ambient air can change considerably depending on the weather. There is no problem with the first three quantities, because when their maximum deviation is known, an estimate for their standard uncertainty can be found (see Table 2) and the uncertainty contributions can be calculated. In the case of the fourth quantity (the inclination angle) the standard uncertainty can also be found, but there will be no contribution to the uncertainty budget in this case because of the zero-value sensitivity coefficient (last row in Table 2).

Table 2. Sensitivity coefficients and uncertainty contributions for input quantities from Table 1 (examples for $M = 1000$ N·m).

Quantity	Standard uncertainty	Sensitivity coefficient	Uncertainty contribution
g	5 µm/s^2	102 N·s^2	0.51·10^{-3} N·m
l	3.5 µm	2000 N	7.00·10^{-3} N·m
ρ_{air}	0.0462 kg/m^3	-0.13 N·m^4/kg	-5.80·10^{-3} N·m
α	1.25 mrad	0 N·m/rad	0.00·10^{-3} N·m

2.1. Comparison of model functions

In order to investigate the situation and to find a way to deal with the sensitivity coefficient of the cosine function near its maximum or minimum values (this also applies to the sine function, respectively), different simple model functions of one variable were investigated.

In the case of a linear function (see Fig. 1), the uncertainty u_x of the input quantity x is “translated” into the uncertainty contribution u_y to the output quantity y just by multiplying u_x with the slope (or gradient) c of the linear function, i.e.

$$u_y = c \cdot u_x \,. \tag{4}$$

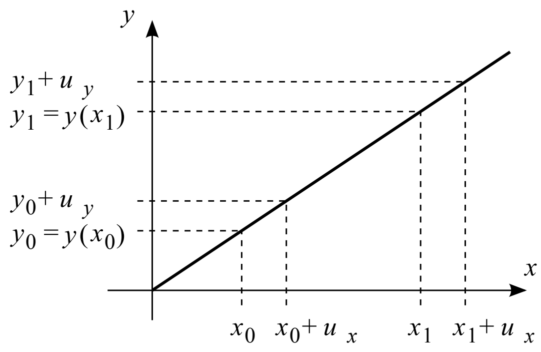

Fig. 1. Uncertainty propagation of a linear function.

The slope is constant, i.e. does not depend on x and can be found from

$$c = (y_1 - y_0)/(x_1 - x_0)\,. \tag{5}$$

This is also the value of the derivative of the given function and the sensitivity coefficient for the input quantity.

In the case of a non-linear function (see Fig. 2), the uncertainty u_x of the input quantity x is translated into the uncertainty contribution u_y to the output quantity y by multiplying u_x with the value of the derivative of the function in the given point, i.e.

$$u_y(x_i) = c(x_i) \cdot u_x \,. \tag{6}$$

In general, u_x can also depend on x_i, resulting in

$$u_y(x_i) = c(x_i) \cdot u_x(x_i)\,. \tag{7}$$

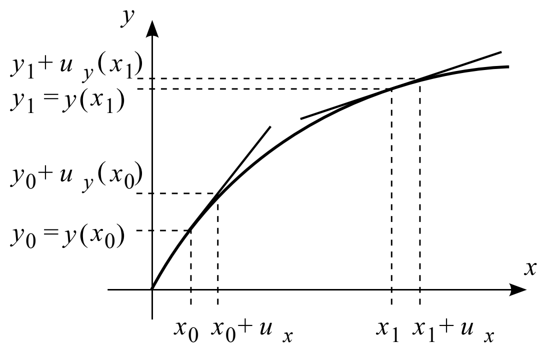

Fig. 2. Uncertainty propagation of a non-linear function.

The value of the derivative, hence, the sensitivity coefficient for the input quantity x, is not constant here. It must be calculated in the given point using the function $y = y(x)$. In the general case of multiple variables

$$y = y(x_1, x_2, x_3, \ldots)\,, \tag{8}$$

the partial derivatives must be taken.

This discussion can be applied to the non-linear, periodic functions cosine and sine (see Fig. 3). In the points where these functions reach their maximum or minimum values, the value of the derivative, i.e. the sensitivity coefficient, is zero and there is no contribution of the input uncertainty u_x to the uncertainty budget of y: $u_y(x_0) = 0$.

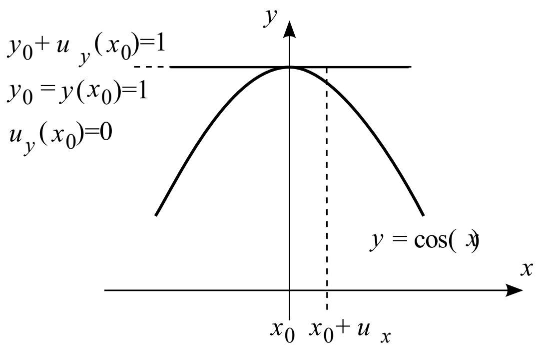

Fig. 3. Uncertainty propagation of a cosine function near zero.

2.2. Proposal for the cosine function with zero estimate

As mentioned before, the problem investigated here is a more common one: the starting point was just the question of how to deal with the sensitivity coefficient in the case of a cosine function near zero, but the problem arises every time there is a function, the derivative of which is zero in the given point. For the periodic functions cosine and sine, these are the values $x_n = n \cdot \pi$, or respectively $x_n = n \cdot \pi + \pi / 2$, with an integer n.

A method is now proposed, which allows a non-zero sensitivity coefficient to be determined (see Fig. 4). The main idea is to approximate the given cosine function by a linear function in the vicinity of the values x_n. For this purpose, the tangent line to the function at the point defined by the uncertainty u_x of the input quantity x is used. In Fig. 4 this line has the designation $t(x)$. It is now shifted in a parallel way so that it runs through the point (x_n; 1) for a maximum or (x_n; -1) for a minimum of the given function. This new line is referred to as $s(x)$ – it is a secant line, intersecting the curve of the function locally in two points.

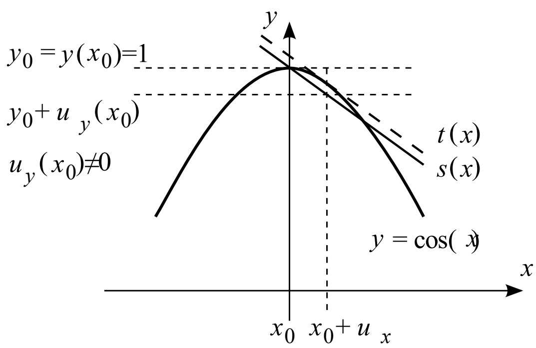

Fig. 4. Proposal for the uncertainty propagation of a cosine function near zero.

We now take the slope of this line $s(x)$ to translate the uncertainty u_x of the input quantity x into the uncertainty contribution u_y to the output quantity y. The slope of the secant as well as that of the tangent is the value of the derivative in the given point, and this is the sensitivity coefficient. In the case of a cosine function this is a sine function and vice versa. The algebraic sign was not taken into account here, because in Fig. 4 the value x_0 - u_x could be considered as well with a positive value of the derivative in this point. In addition, the single uncertainty contributions are squared when the overall uncertainty is calculated, so that a negative sign would disappear.

In the special case of a cosine function, the uncertainty contribution u_y and the sensitivity coefficient $c(u_x)$can be found from

$$u_y = -\sin(u_x) \cdot u_x \text{ and} \tag{9}$$

$$c(u_x) = -\sin(u_x)\,. \tag{10}$$

In contrast to the case of a non-linear function that was discussed above, the sensitivity coefficient is now not a function of the estimate x_i, but it is a function of the uncertainty u_x of this estimate.

From Fig. 4 it can be seen that the contribution of u_x to the uncertainty of y is overestimated: for a given x within the interval (x_0; $x_0 + u_x$), the distance of the line $s(x)$ from the constant function $y = 1$ exceeds this distance for the cosine function. But for very small values of u_x the effect is negligible.

2.3. Application to the torque uncertainty budget

When the method described above is applied to the uncertainty budget for the torque, a sensitivity coefficient of -1.25 N·m/rad for the inclination angle α is obtained (see last row in Table 3). The standard uncertainty is quite large, it corresponds to a ratio of 0.5 mm inclination uncertainty for a base length of 400 mm. This value is rather an upper limit, much smaller values can be realized quite easily. The uncertainty contribution of $-1.6 \cdot 10^{-3}$ N·m in this case is comparable to the other contributions. Another advantage of the method is the possibility to derive a value for the standard uncertainty for a given uncertainty contribution. In other words, one can estimate, just how accurately the inclinations must be measured in order not to exceed an uncertainty contribution of, for example, 1 mN·m.

Table 3. Enhanced sensitivity coefficient and uncertainty contributions for α (examples for M = 1000 N·m).

Quantity	Standard uncertainty	Sensitivity coefficient	Uncertainty contribution
α	1.25 mrad	-1.25 N·m/rad	$-1.60 \cdot 10^{-3}$ N·m

3. CONCLUSION

The method proposed in this paper allows the uncertainty contribution of an input quantity to be estimated in cases where the derivative of a model's cosine or sine function and, hence, the sensitivity coefficient calculated according to the standard GUM procedure, is zero.

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