UNCERTAINTY AND INTERPOLATION

Liron Anavy, Eng. Boris Katz

Quality Control Center Hazorea, Israel

ABSTRACT

This article deals with interpolation methods, ways to determine which method is required and ways to check the results' fitness.

1. INTRODUCTION

In order to minimize uncertainty of calibration of force transducers it became normal to use an interpolation. Usually the linear interpolation is taken, though it's the least effective method [1] it's the easiest to compute. But when approaching a higher-level polynoms it might get a bit more problematic to calculate the correct coefficients. The most obvious problem, which we encountered as soon as we started to use interpolation, is "why linear interpolation in some cases is better than 2nd or 3rd degree polynom?"

2. MODELS

The answer is – we use different methods for the interpolation error and the results comparison. When we use the "Least Sum Squares" method we're taking

$$F \approx f(R) = a_0 + a_1 R^1 + .. + a_N R^N \tag{1}$$

Assuming there exists only one minimum for each coefficient we get the following set of linear equations

$$\begin{aligned}
&\frac{\partial}{\partial a_i}\sum [F - f]^2 = 0 \\
&\frac{\partial}{\partial a_i}\sum [F - f]^2 = \sum 2[F - f]\frac{\partial}{\partial a_i} f = \sum 2[F - f]R^i \Rightarrow \\
&\Rightarrow \sum \left[FR^i - a_0 R^i - a_1 R^{i+1} .. a_N R^{i+N}\right] = 0
\end{aligned} \tag{2}$$

The resulting matrix is

$$\begin{matrix}
n_R & \sum R & \rightarrow & \sum R^N & \sum F \\
\sum R & \sum R^2 & \rightarrow & \sum R^{N+1} & \sum FR \\
\downarrow & \downarrow & & \downarrow & \downarrow \\
\sum R^N & \sum R^{1+N} & \rightarrow & \sum R^{2N} & \sum FR^N
\end{matrix} \tag{3}$$

The matrix can be solved easily by using Gauss method.

The problem is that in (2) we "landed" on the absolute deviation minima. In the case we want the F.S deviation to be minimal it's the correct method, but, in the case we want the O.R deviation to be minimal the correct set would be:

$$\frac{\partial}{\partial a_i}\sum\left[\frac{F-f}{F}\right]^2 = 0$$

$$\frac{\partial}{\partial a_i}\sum\left[\frac{F-f}{F}\right]^2 = \sum 2\left[\frac{F-f}{F}\right]\frac{\partial}{\partial a_i}\frac{f}{F} = \sum 2\left[\frac{F-f}{F}\right]\frac{R^i}{F} \Rightarrow \tag{4}$$

$$\Rightarrow \sum\left[\frac{FR^i}{F^2} - \frac{a_0 R^i}{F^2} - \frac{a_1 R^{i+1}}{F^2} .. \frac{a_N R^{i+N}}{F^2}\right] = 0$$

And the matrix for the Gauss:

$$\begin{matrix} \sum\frac{1}{F^2} & \sum\frac{R}{F^2} & \rightarrow & \sum\frac{R^N}{F^2} & \sum\frac{1}{F} \\ \sum\frac{R}{F^2} & \sum\frac{R^2}{F^2} & \rightarrow & \sum\frac{R^{N+1}}{F^2} & \sum\frac{R}{F} \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \sum\frac{R^N}{F^2} & \sum\frac{R^{1+N}}{F^2} & \rightarrow & \sum\frac{R^{2N}}{F^2} & \sum\frac{R^N}{F} \end{matrix} \tag{5}$$

The solutions of (3) and (5) are not the same for most cases, actually unless the interpolation error is 0 for all measurements it can't be the same.

3. FITNESS CHECKING

For the fitness checking one can use a Gaussian

$$e^{-\frac{1}{2}\chi^2} \quad \chi^2 \equiv \sum(F-f)^2 \tag{6}$$

Multiplying (6) with the measurements' spreading gives

$$e^{-\frac{1}{2}\left[\chi^2+\sigma^2\right]} \tag{7}$$

This function determines the goodness of fit and function can be interpreted to tell the probability of correct fitting; the exact number has to be found using experiments. Unlike the method described in [2] this method does not require any additional calculations, and can give an accurate verification immediately.

Measurement's uncertainty can be added in the same way, but since we prefer the absolute error for each reading's interpolation for use in the "Root Sum Squares" we haven't gone into this adventure.

4. EXTENSIONS TO THE MODELS

It can be easily verified that this model acts the same on a complex variable (x), thus giving us a way to couple 2 variables:

$$\begin{aligned} x_1 &= \mathrm{Re}(z) \\ x_2 &= \mathrm{Im}(z) \end{aligned} \tag{8}$$

When using that method one must remember that the coefficients are also complex.

Another possible extension, which can be used for certain calculations, is:

$$\frac{\partial}{\partial a_i}\sum\left[\frac{F-f}{F}\right]^2 = 0 \Rightarrow \sum\left[\frac{R^i}{F} - \frac{a_0 R^i f_0}{F^2} - \frac{a_1 R^i f_1}{F^2} .. \frac{a_N R^i f_N}{F^2}\right] = 0 \tag{9}$$

Replacing f_j with R^j gave us the “regular” form, but in some cases it would be better to use another function set.

5. USAGE

Once the tools are established we need to check their efficiency. For this matter we’ve used some of our force-transducers, the results are:

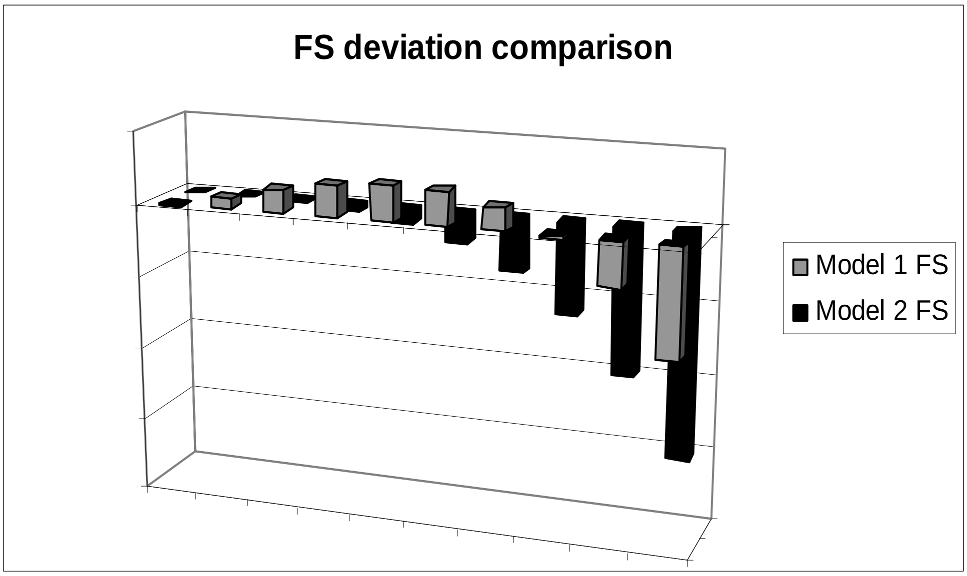

Figure 1: Full-scale deviation of the models from the measured values**.**

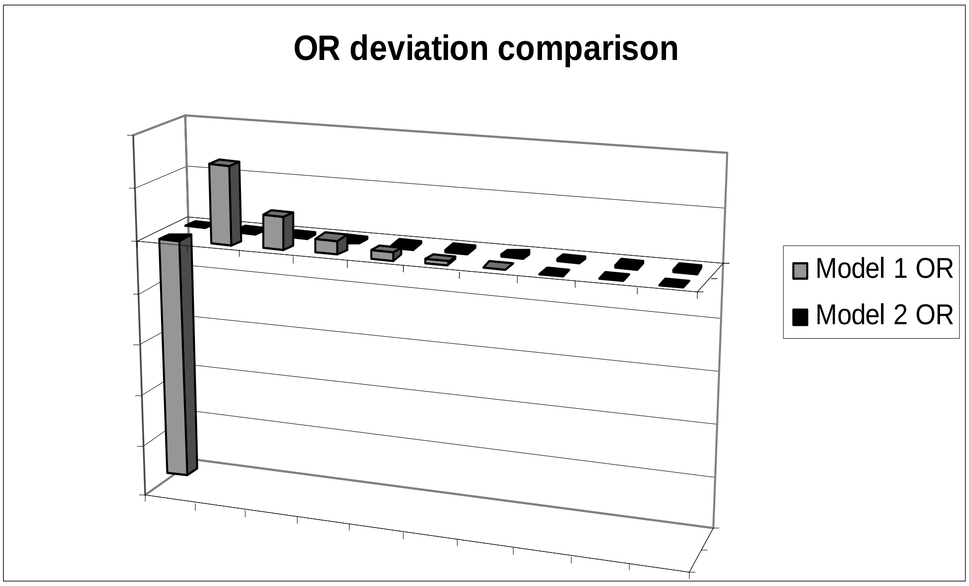

Figure 2 – of-reading deviation of the models from the measured values.

In figure 1 the deviations of the 2nd model increases while the 1st model has more or less the same magnitude for all values. In figure 2 the 1st model decreases while the 2nd has the same magnitude.
In order to draw the graphs we used a linear approximation to a higher degree polynom. A different function would have the same general behavior, though it might not be as obvious as figures 1 and 2.
Note: since the minima condition is applied to the summary the maximum deviation can be higher in the more suitable model, it would be advisable to use a statistical method [3] in order to calculate the uncertainty in those cases.

6. UNCERTAINTY OF THE MODELS

The required effective digits for calculation is

$$D \times (P+1) \tag{10}$$

Where D is the actual required digits, and P is the degree of the polynom. Using insufficient digits in the calculation can make the higher degrees a good source for uncertainties, unless their related coefficient is small enough for the rounded part to vanish.
For example I'll take a 0.001% uncertainty on a 64bit machine will require:

$$\begin{aligned} D &= Log_2(0.001\%) = 17 \\ {}^{64}\!/_{17} &= 3 \Rightarrow P_{eff} = 2 \end{aligned} \tag{11}$$

Note that the value of D must be rounded up, while the value of P must be rounded down. An 80bit machine would have P=3.
The errors of the calculation arise from the rounded digits and are multiplied by the relevant coefficient giving

$$\sum_{i>P_{eff}} \left[D \times (i+1) - M_D\right] \cdot a_i \qquad (12)$$

Where P_{eff} is the highest “error free” degree of the polynom.
Since this error has a normal distribution and no correlations it can be added to a Root Sum Squares formula without any problems.

7. CONCLUSIONS

An interpolation model should be made for the correct usage, i.e. the model that gives best results for a certain type does not necessarily gives good results for another type (F.S and O.R are the most common example)

A polynom of 2^{nd} degree can be easily used without the need to add anything to the calculated uncertainty on any new computer (hardware only), a 3^{rd} degree or higher should be added to the uncertainty calculation.

We suggest that [2] would also apply some modifications to the interpolation usage in order to take advantage of newer (faster) computers and to use an uncertainty model based on the fitness.

REFERENCES

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3. Parasenjit Saha Principles of Data Analysis, Queen Mary and Westfield College, University of London (Cappella Archive)

Addresses of the Authors:

Liron Anavy Quality Control Center Hazorea, Kibbutz Hazorea 30060, Israel
liron@mba.co.il

Eng. Boris Katz Head of Force, Torque and Hardness Department, Quality Control Center Hazorea, Kibbutz Hazorea 30060, Israel
boris_ka@mba.co.il