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Measurement Uncertainties and Efficiencies Using Correlated and Un-correlated Cycles of Comparison Measurements

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Abstract

Comparison measurements between a standard and a test object may be performed and evaluated by various sequences of measurement cycles in repetition. It is understood that from each cycle a single measurement result and from the total number of them a mean and an uncertainty are evaluated. Consecutive cycles may have independent data or a certain number of common data. When evaluating of the measurement uncertainty of the mean, cycles with common data have to be considered as correlated and corresponding procedures have to be observed, which differ from the statistical procedure for a number of independent observations. Here, the uncertainty calculation for the often-used correlated RTR cycle is presented followed by a calculation for any type of correlated or uncorrelated cycle. Finally, an efficiency is defined and discussed for some selected cycles.

1. Introduction

The uncertainty of the result of a measurement consists of different contributions [1], at least of that of the value of the standard and of that of the measured difference value between test object T and standard R. Only the value of this difference, $\Delta x = x_T - x_R$, and its uncertainty contribution shall be the subject of this paper. For evaluating a statistically based uncertainty, Δx has to be measured repeatedly. There are different possible procedures for repeated measurements of x_T and x_R and also for the evaluation of Δx from the measurements. A basic requirement for the procedure is the elimination of drifts occurring with time.

Therefore, a symmetric sequence of T and R, called cycle, is preferred for the evaluation of a single value of Δx. A cycle may have the sequence RTTR, TRRT, RTR, TRT or others. For statistical reasons, repeated cycles shall have the same sequence. Consecutive cycles are statistically independent of each other, if complete cycles are repeated, but they are not independent, if consecutive cycles have common data. For example, if in a measurement sequence $R_1T_1R_2T_2R_3T_3R_4$ ... the cycle $R_1T_1R_2$ gives the first Δx value, $R_2T_2R_3$ the second, $R_3T_3R_4$ the third, each cycle has a R that is common with the following or the previous one . In the uncertainty calculation with non-independent cycles, correlations between consecutive cycles have therefore to be taken into account (see also [2]).

Considering different measurement sequences, independent and non-independent cycles, the question may arise, which is the best procedure, that means which cycle gives the smallest measurement uncertainty for the same number of single measurements of R and T. A quantitative measure of this property will be defined as an efficiency with a maximum value of one for the best solution.

2. Correlated RTR cycles

The sequence of measurements and their numbering is defined as follows:

$R_1T_1R_2T_2R_3T_3$...

Value of the i-th difference:

$$\Delta x_i = x_{\mathrm{T}i} - \frac{x_{\mathrm{R}i} + x_{\mathrm{R}i+1}}{2} \tag{1}$$

The variance of single measurements is assumed to be the same for R and T, because R and T are usually similar objects and the measurement procedure is essentially the same:

$$\sigma^2(x_{\mathrm{T}i}) = \sigma^2(x_{\mathrm{R}i}) = \sigma^2(x_{\mathrm{R}i+1}) =: \sigma^2(x). \tag{2}$$

variance of the i-th difference, given in terms of $\sigma^2(x)$:

$$\sigma^2(\Delta x_i) = \frac{3}{2}\sigma^2(x) \tag{3}$$

covariance between consecutive differences:

$$\mathrm{cov}(\Delta x_i, \Delta x_{i+1}) = \frac{\partial \Delta x_i}{\partial x_{\mathrm{R}i+1}} \frac{\partial \Delta x_{i+1}}{\partial x_{\mathrm{R}i+1}} \sigma^2(x_{\mathrm{R}i+1})$$

$$= \frac{1}{4}\sigma^2(x), \tag{4}$$

variance of the mean:

$$\sigma^2(\overline{\Delta x}) = \sigma^2\left(\frac{1}{n}\sum_{i=1}^{n} \Delta x_i\right) \tag{5}$$

or:

$$\sigma^2(\overline{\Delta x}) = \frac{1}{n^2}\left[\sum_{i=1}^{n} \sigma^2(\Delta x_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathrm{cov}(\Delta x_i, \Delta x_j)\right] \tag{6}$$

see eq.(3) and eq.(4)

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n} \frac{3}{2}\sigma^2(x) + 2\sum_{i=1}^{n-1} \frac{1}{4}\sigma^2(x)\right] \tag{7}$$

$$\sigma^2(\overline{\Delta x}) = \frac{4n-1}{2n^2}\sigma^2(x). \tag{8}$$

From measurements, the experimental variance of Δx is determined as:

$$s^2(\Delta x) = \frac{1}{n-1}\sum_{i=1}^{n} (\Delta x_i - \overline{\Delta x})^2 \tag{9}$$

For estimating $\sigma(\overline{\Delta x})$ from eq.(8) we need therefore a relation between $s(\Delta x)$ and $s(\overline{\Delta x})$, the estimator of $\sigma(\overline{\Delta x})$. In cases, where the Δx_i are independent of each other, that is un-correlated, we have:

$$s^2(\overline{\Delta x}) = \frac{s^2(\Delta x)}{n}. \tag{10}$$

In cases, where they are correlated, eq.(10) is no more valid. A general consideration on variances will lead to the correct relation. The variances of Δx and $\overline{\Delta x}$ are defined as:

$$\sigma^2(\Delta x) = E\left[(\Delta x - \mu)^2\right]$$
$$\sigma^2(\overline{\Delta x}) = E\left[(\overline{\Delta x} - \mu)^2\right] \qquad (11)$$

where E is the expectation of the value given in brackets and μ is the expectation of Δx or $\overline{\Delta x}$. Under the assumption that Δx and $\overline{\Delta x}$ are randomly distributed, we have:

$$E(\Delta x - \mu) = 0 \qquad \text{and}$$
$$E(\overline{\Delta x} - \mu) = 0 \qquad (12)$$

If n measurement data of Δx are available, we may write:

$$\sigma^2(\Delta x) = E\left[\left(\frac{1}{n}\sum_{i=1}^{n}(\Delta x_i - \mu)\right)^2\right] \qquad (13)$$
$$= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}\left((\Delta x_i - \overline{\Delta x}) - (\mu - \overline{\Delta x})\right)\right)^2\right]$$

$$= \frac{1}{n}E\left[\sum_{i=1}^{n}(\Delta x_i - \overline{\Delta x})^2\right] - 2E\left[\frac{1}{n}\sum_{i=1}^{n}(\Delta x_i - \overline{\Delta x})(\mu - \overline{\Delta x})\right] + E\left[(\mu - \overline{\Delta x})^2\right]$$

$$\sigma^2(x) = \frac{1}{n}E\left[\sum_{i=1}^{n}(\Delta x_i - \overline{\Delta x})^2\right] + \sigma^2(\overline{\Delta x})$$

see eq.(11) and eq.(12)

$$E\left[\sum_{i=1}^{n}(\Delta x_i - \overline{\Delta x})^2\right] = n\left[\sigma^2(\Delta x) - \sigma^2(\overline{\Delta x})\right] \qquad (14)$$

The left hand side of eq.(14) is the expectation of $(n-1)s^2(\Delta x)$, which is calculated by using eq.(9) – independent of if the Δx_i are correlated or not.

$$E\left[s^2(\Delta x)\right] = \frac{n}{n-1}\left[\sigma^2(\Delta x) - \sigma^2(\overline{\Delta x})\right] \qquad (15)$$

The right hand side of eq.(15) contains the variances of Δx and $\overline{\Delta x}$, which are understood as based on un-correlated data - by definition. We now apply eq.(15) to the RTR cycles using eq.s (3) and (8) and come to the following relations:

$$E\left(s^2(\Delta x)\right) = \frac{n}{n-1}\left[\frac{3}{2}\sigma^2(x) - \frac{4n-1}{2n^2}\sigma^2(x)\right]$$

$$\sigma^2(\overline{\Delta x}) = \frac{4n-1}{n(3n-1)}E\left(s^2(\Delta x)\right)$$

see eq,(8) (16)

or in terms of estimators:

$$u(\overline{\Delta x}) = \sqrt{\frac{4n-1}{3n-1}}\,\frac{s(\Delta x)}{\sqrt{n}} \qquad (17)$$

Equation (17) gives the experimental standard uncertainty of the mean, that is the standard uncertainty of Δx as a function of the experimental standard deviation of Δx for correlated cycles RTR.

3. Any cycle

Generalizing the problem, we consider now any correlated or uncorrelated measurement cycle and use a new numbering system of the single measurements of R and T:

$$\Delta x_1\left(x_{\mathrm{T},1}, x_{\mathrm{T},2}, \ldots, x_{\mathrm{T},t}, x_{\mathrm{R},1}, x_{\mathrm{R},2}, \ldots, x_{\mathrm{R},r}\right)$$

$$\Delta x_2\left(x_{\mathrm{T},p+1}, x_{\mathrm{T},p+2}, \ldots, x_{\mathrm{T},p+t}, x_{\mathrm{R},q+1}, x_{\mathrm{R},q+2}, \ldots, x_{\mathrm{R},q+r}\right)$$

$$\Delta x_n\begin{pmatrix} x_{\mathrm{T},(n-1)p+1}, x_{\mathrm{T},(n-1)p+2}, \ldots, x_{\mathrm{T},(n-1)p+t}, x_{\mathrm{R},(n-1)q+1}, \\ x_{\mathrm{R},(n-1)q+2}, \ldots, x_{\mathrm{R},(n-1)q+r} \end{pmatrix} \quad (18)$$

For a single difference, we have:

$$\Delta x_i = \frac{1}{t}\sum_{j=1}^{t} x_{\mathrm{T},(i-1)p+j} - \frac{1}{r}\sum_{k=1}^{r} x_{\mathrm{R},(i-1)q+k} \quad (19)$$

$$\frac{\partial \Delta x_i}{\partial x_{\mathrm{T},(i-1)p+j}} = \frac{1}{t}\,; \quad \frac{\partial \Delta x_i}{\partial x_{\mathrm{R},(i-1)q+k}} = -\frac{1}{r} \quad (20)$$

In a similar way as for the correlated RTR cycle, we obtain:

$$\sigma^2\left(\overline{\Delta x}\right) = \frac{1}{n^2}\left[\sum_{i=1}^{n}\left[\sum_{j=1}^{t}\left[\frac{1}{t}\right]^2 + \sum_{k=1}^{r}\left[\frac{1}{r}\right]^2\right]\sigma^2(x) + 2\sum_{i=1}^{n-1}\sum_{l=i+1}^{n}\operatorname{cov}\left(\Delta x_i, \Delta x_l\right)\right]$$

(21)

We define $t^* = \operatorname{Int}\left(\frac{t-1}{p}\right)$ and $r^* = \operatorname{Int}\left(\frac{r-1}{q}\right)$ as integer numbers equal to the fractions in brackets or down rounded in case these fractions are not integer.

$$\sigma^2\left(\overline{\Delta x}\right) = \frac{1}{n^2}\left[n\left(\frac{1}{t}+\frac{1}{r}\right)\sigma^2(x) + 2\sum_{i=1}^{n-1}\left[\sum_{l=i+1}^{i+t^*}\operatorname{cov}\left(\Delta x_i(x_\mathrm{T}), \Delta x_l(x_\mathrm{T})\right) + \sum_{l=i+1}^{i+r^*}\operatorname{cov}\left(\Delta x_i(x_\mathrm{R}), \Delta x_l(x_\mathrm{R})\right)\right]\right]$$

(22)

$$\operatorname{cov}\left(\Delta x_i(x_\mathrm{T}), \Delta x_l(x_\mathrm{T})\right) = \left[\sum_{j=(l-i)p+1}^{t}\frac{\partial \Delta x_i}{\partial x_{\mathrm{T},(i-1)p+j}}\,\frac{\partial \Delta x_l}{\partial x_{\mathrm{T},(i-1)p+j}}\right]\sigma^2(x)$$

$$\operatorname{cov}\left(\Delta x_i(x_\mathrm{T}), \Delta x_l(x_\mathrm{T})\right) = \left(t-(l-i)p\right)\frac{1}{t^2}\sigma^2(x)$$

see eq.(20) (23)

For the other covariance we find:

$$\operatorname{cov}\left(\Delta x_i(x_\mathrm{R}), \Delta x_l(x_\mathrm{R})\right) = \left(r-(l-i)q\right)\frac{1}{r^2}\sigma^2(x)$$

(24)

and:

$$\sum_{l=i+1}^{i+t^*}\operatorname{cov}\left(\Delta x_i(x_\mathrm{T}), \Delta x_l(x_\mathrm{T})\right)$$

$$= \frac{t^*}{t}\left[1-\frac{(t^*+1)p}{2t}\right]\sigma^2(x) \quad (25)$$

With a similar derivation of the other sum of eq.(22) we have:

$$\sigma^2\left(\overline{\Delta x}\right) = \frac{1}{n}\left[\frac{1}{t}+\frac{1}{r}+2\frac{n-1}{n}A\right]\sigma^2(x) \quad (26)$$

with:

$$A = \frac{t^*}{t}\left[1-\frac{(t^*+1)p}{2t}\right] + \frac{r^*}{r}\left[1-\frac{(r^*+1)q}{2r}\right] \quad (27)$$

From eq.(15) follows the estimate:

$$u\left(\overline{\Delta x}\right) = \sqrt{\frac{n(t+r)+2Art(n-1)}{n(t+r)-2Art}}\,\frac{s(\Delta x)}{\sqrt{n}}$$

for: $n(t+r) > 2Art$ (28)

Equation (28) gives the standard uncertainty of a cycle with t values of T and r values of R. The factor A considers correlations between consecutive cycles - it vanishes for un-correlated ones.

We may now define an enhancing factor c, by which the uncertainty for correlated cycles differ from an uncertainty, which disregards the correlation:

$$c = \sqrt{\frac{n(t+r)+2Art(n-1)}{n(t+r)-2Art}} \qquad (29)$$

4. Efficiency

For the question, which cycle requires a minimum number of measurements for a required uncertainty, an efficiency is defined as follows:

$$\eta = \frac{4\sigma^2(x)}{n_S u^2(\Delta x)} \qquad (30)$$

n_S total number of single measurements.

Figure 1 shows the efficiencies of some selected cycles as a function of the number of single measurements. It shows clearly, that the most efficient cycle is the un-correlated RTTR cycl (r - q = 0). The efficiencies of the un-correlated cycles are independent of the number of measurements and they are closer to η = 1 the closer their numbers of R's and T's are.

Another example gives different possible evaluations of a RTR...R sequence. Certain numbers n_S of single measurements may be divided into integer numbers n of different cycles. Table 1 shows the efficiencies and the enhancing factors for three sequences. Among these examples, the overlapping RTR cycle has the largest efficiency, which can also be seen form Figure 1. The enhancing factors show, to which extent the uncertainty would be underestimated, if correlations would not have been considered.

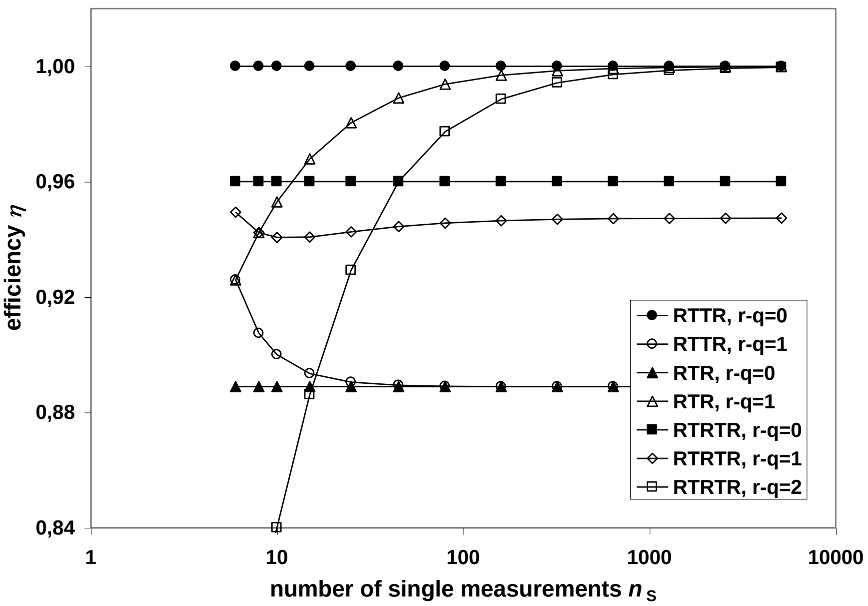

Figure 1. Efficiency η as a function of the total number n_S of single measurements R or T, shown for some selected cycles. *r-q* gives the degree of overlapping (*r-q*=0 is not overlapping, the cycles are uncorrelated)..

Table 1. Efficiencies η and enhancing factors c, given for two cycles with different degrees of overlapping (r - q). Three total numbers of single measurements n_S of sequences RTR...R, giving corresponding integer numbers n of the respective cycles arrangements, are considered.

		n_S = 15			n_S = 45			n_S = 105		
Cycle	***r* - *q***	***n***	η	***c***	***n***	η	***c***	***n***	η	***c***
RTR	0	5	0,8889	1	15	0,8889	1	35	0,8889	1
RTR	1	7	0,9679	1,1619	22	0,989	1,1569	52	0,9953	1,1556
RTRTR	0	3	0,96	1	9	0,96	1	21	0,96	1
RTRTR	1	-	-	-	11	0,9444	1,1284	26	0,9460	1,1267
RTRTR	2	6	0,8862	1,6811	21	0,96	1,5811	51	0,9827	1,5619

5. References

[1]. *Guide to the Expression of Uncertainties in Measurement*, International Organization of Standardization, Geneva, 1993

[2]. Gläser, M., *Cycles of comparison measurements, uncertainties and efficiencies*, Meas. Sci. Technol., 2000, **11**, p. 20-24

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