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Precise Measurand Value Estimating by Interval Fusion with Preference Aggregation: Heteroscedasticity Case

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Abstract **– It is considered the problem of determination of a reference value for heteroscedastic interval data. For this aim, the newly proposed by the authors interval fusion with preference aggregation (IF&PA) procedure is used. The procedure, modified to improve the accuracy of the fusion result, is presented and applied to process the heteroscedastic data of a real experiment. The experiment consisted in determination of the reference value of DC voltage based on the readings of five different models of multimeters. For comparison, the same data were processed by the method of weighted mean. For two methods, an absolute deviations of the obtained reference values from the nominal value (which is a high-precision calibrator output) and reference value uncertainties were estimated. It is shown that the IF&PA procedure allows to obtain a reference value very close to nominal value and with considerably reduced uncertainty in comparison with traditional method based on weighted mean calculation.**

Keywords – Interval fusion, Preference aggregation, Heteroscedasticity, Consensus estimate, Ranking.

I. INTRODUCTION

Statistical methods for processing measurement data have specific limitations imposed on permissible properties of data, such as normality of the probability distributions, independence of observations, absence of outliers, etc. However, in real conditions the measured values are often heteroscedastic, i.e. characterized by unequal dispersion. It usually has place when measuring a certain quantity by different instruments and (or) by various measurement methods, or under dissimilar environmental conditions [1]. Heteroscedasticity of measurement data occurs, in particular, in such situations as: interlaboratory and (or) key comparisons [2, 3], adjustment of the values of fundamental constants [4], data collection from nodes of sensor networks [5, 6], etc. [7-9]. Heteroscedasticity can have serious effect on the results of data processing, so it should be taken into account and appropriately managed [10].

Due to their inherent properties, most parametric statistic methods cannot deal with heteroscedastic data effectively enough without preliminary consistency check and outliers elimination. In such case, it is often recommended to apply non-parametric methods which require fewer assumptions for the data [3]. However, though the non-parametric methods can give a good result for abnormal data, they can hardly be used when the number of measured values is small. Another controversial issue is outliers' detection as the corresponding procedures applied to heteroscedastic data can lead to unnecessary data removing.

Earlier, the authors have published a series of papers [11–16] devoted to so called interval fusion with preference aggregation (IF&PA) technique. The authors believe that the proposed method has promising potential for use in processing heteroscedastic measurement data. In papers [14–16], through numerical experiments, an accuracy and robustness of the interval fusion result (reference value) obtained at the output of the IF&PA procedure was investigated. In terms of these indicators, the IF&PA procedure, as a rule, surpasses the traditional method of processing heteroskedastic data, i.e. the weighted mean calculation.

In this paper, the IF&PA procedure, modified to improve the accuracy of the fusion result, is used to process the data of a real experiment consisting in determination of the reference value of DC voltage based on the readings of five different models of multimeters.

II. IF&PA: INTERVAL FUSION WITH PREFERENCE AGGREGATION

By *interval data fusion* we understand a procedure of determination of a resulting interval $[x^* \pm u^*]$ which is consistent with the maximal number of initial intervals $\{I_k\}$, $k = 1, \ldots, m$, and with maximum likelihood contains a point x^* (fusion result) that can be considered as a representative of all the intervals $\{I_k\}$ with an uncertainty u^* [15].

A value x_k of the measurand along with its uncertainty u_k is represented as an *interval* $I_k = [x_k^l, x_k^u]$ on the real line which is defined by its lower bound x_k^l, upper bound x_k^u and middle point x_k. Let us consider a set of initial intervals $\{I_k\}$, $k = 1, \ldots, m$, as input data for the IF&PA method .

The IF&PA method consists of the following three steps as described below.

Step1. Forming the range of actual values (RAV) $A = \{a_1, a_2, ..., a_n\}$ by the union of initial intervals $\{I_k\}$, $k = 1, \ldots, m, \}$, where the least lower bound x_k^l is selected to be the lower bound of the RAV, i.e. $a_1 = \min\{x_k^l \mid k = 1, ..., m\}$, and the largest upper bound x_k^u is chosen to be the upper bound of RAV, i.e. $a_n = \max\{x_k^u \mid k = 1, ..., m\}$. The united interval is partitioned into $n - 1$ equal subintervals of length (or *norm*) $h = (a_n - a_1) / (n - 1)$ to obtain discrete elements $a_2, a_3, \ldots, a_{n-1}$. The appropriate number n, or *cardinality* of the RAV partition, can be selected beforehand [15,16].

Step 2. Representing intervals by inrankings, i.e rankings λ_k induced by intervals $\{I_k\}$, composed according to the following conditions for $i, j = 1, \ldots, n$:

$$\begin{gathered}\text{(i) } a_i \in I_k \wedge a_j \notin I_k \Rightarrow a_i \succ a_j;\\ \text{(ii) } a_i, a_j \in I_k \vee a_i, a_j \notin I_k \Rightarrow a_i \sim a_j;\\ \text{(iii) } a_i \notin I_k \wedge a_j \in I_k \Rightarrow a_i \prec a_j;\\ \text{(iv) } a_i, a_j \in I_k \text{ are neighbors} \Rightarrow j \equiv i + 1,\end{gathered} \tag{1}$$

where two types of binary relations are used: *strict order* $a_i \succ a_j$ and *tolerance* $a_i \sim a_j$. Composing the input preference profile $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\}$ of the m inrankings λ_k.

Step 3. Aggregating inrankings that is determining a fusion result x^* as the most preferable alternative in consensus ranking β_{fin} for the input profile Λ. All possible consensus rankings are found by recursive algorithm RECURSALL implementing the *Kemeny rule* [17] and forming the output profile $B(N, n) = \{\beta_1, \beta_2, \ldots, \beta_N\}$ consisting of multiple consensus rankings. In order to transform the output profile $B(N, n)$ into a single final consensus ranking β_{fin}, the following *convolution rule* is applied [18]. Let *rank* r_i^k be a position of an alternative a_i in the consensus ranking $\beta_k \in B$, $k = 1, \ldots, N$. Let a *total rank* r_i of the alternative a_i is defined as $r_i = \sum_{k=1}^{N} r_i^k$. Then, for all $i < j$, $i, j = 1, \ldots, n$,

$$r_i < r_j \Rightarrow a_i \succ a_j \text{ and } r_i = r_j \Rightarrow a_i \sim a_j, \tag{2}$$

where both of the relations $\succ$ and $\sim$ are in the single final consensus ranking β_{fin}.

The most preferable alternative a_i which takes the leftmost position in the obtained ranking β_{fin} is assigned to be a fusion result x^*. It means that the rank r_i of this alternative equals to 1, i.e.

$$x^* = a_i^1,\ a_i^1 \in \beta_{\text{fin}}\,. \tag{3}$$

If β_{fin} contains more than one best alternatives, i.e. $\{a_i^1 \sim a_j^1 \sim ... \sim a_k^1\} \subseteq \beta_{\text{fin}}$ for $i, j, k = 1, \ldots, m$, then a natural single representative of all these alternatives (they are already sorted in ascending order) is a *sample median*:

$$x^* = \begin{cases} a^1_{(k+1)/2}, & \text{if } k \text{ is odd;} \\ (a^1_{k/2} + a^1_{1+k/2})/2, & \text{if } k \text{ is even.} \end{cases} \tag{4}$$

The uncertainty u^* of the fusion result x^* depends on the norm h (see Step 1) and can be estimated as follows:

$$u^* = \pm 0.5h. \tag{5}$$

Formula (5), evidently, gives rather rough estimate of the uncertainty u^*, and after Step 3 the IF&PA procedure has comparatively low accuracy.

Therefore, the IF&PA procedure has been modified by adding Steps 4–6 for determination of a more accurate fusion result x^{**}. The modification was implemented by means of the IF&PA reuse for the updated RAV in neighbourhood of x^* within boundaries equal to the initial RAV partition norm $\pm h$. The effect obtained is similar to that achieved using the *vernier scale* [19].

III. FURTHER IF&PA STEPS

The following steps are further implemented to increase the accuracy of the fusion result x^* obtained in the previous stages.

Step 4. Forming the new range of actual values $E = \{e_1, e_2, ..., e_{11}\}$ with the lower bound $e_1 = x^* - 0.5h$ and the upper bound $e_{11} = x^* + 0.5h$, that is the RAV E is partitioned into $n - 1 = 10$ equal subintervals of length (norm) $h' = 0.1h$ to produce elements $e_2, e_3, \ldots, e_{10}$.

Step 5. Representing *updated* initial intervals $\{I_k\}$ by new inrankings ω_k. Notice that the initial intervals $\{I_k\}$ are included into the new RAV E only by those parts that satisfy the condition

$$e_1 \le e_i \le e_{11},\ e_i \in I_k,\ i = 1, \ldots, 11. \tag{6}$$

Therefore, some of the updated initial intervals will be truncated and some will be ignored as not satisfying the condition (6), then the value m can be reduced to m_{con}. Inrankings ω_k are formed using Eq. (1) where instead of A the set $E = \{e_1, e_2, ..., e_{11}\}$ is used and the initial intervals $\{I_k\}$ are updated. Thus, the new preference profile $\Omega = \{\omega_1, \omega_2, ..., \omega_{m_{\text{con}}}\}$ is composed.

Step 6. Determining a more accurate fusion result x^{**} as the best alternative in consensus ranking β'_{fin} for the profile Ω. That is, the same operations as in Step 3 are fulfilled for the new profile Ω instead of Λ.

The uncertainty u^{**} of the fusion result x^{**} is calculated as a half of the partition norm h', and we finally have

$$u^{**} = 0.5h' = 0.05h. \quad (7)$$

IV. EXPERIMENTAL VERIFICATION

To test the improved IF&PA method, DC voltage measurements were performed using five different models of digital multimeters: M838, DT9205A, UT61E, B7-38M and MY-68.The measurement result accuracy of each multimeter was calculated as the value of the maximum permissible error (MPE) according to the formula, which is reported by the manufacturer (see Table 1). The MPE defines boundaries u_k of the interval that characterizes each measurement result x_k. Besides method verification, one of the aims of the experiment was to confirm that the IF&PA can effectively deal with real heteroscedastic data even when a small number of measured values are accessible.

Table 1. Uncertainty specifications of the digital multimeters for DC voltage.

Symbol	Model	Range, V	LSD[1], mV	Absolute accuracy, V
M1	M838	0.2	0.1	± (0.25% + 2LSD)
		2	1	± (0.5% + 2LSD)
		20	10	
		200	100	
M2	DT9205A	0.2	0.1	± (0.5% + 2LSD)
		2.0	1	
		20.0	10	
		200.0	100	
M3	UT61E	0.22	0.01	± (0.1% + 5LSD)
		2.2	0.1	
		22.0	1	± (0.1% + 2LSD)
		220.0	10	
M4	V7-38M	0.3	0.01	± (0.03% + 3LSD)
		3.0	0.1	
		30.0	1	
		300.0	10	
M5	MY-68	0.4	0.1	± (0.7% + 2LSD)
		4.0	1	
		40.0	10	
		400.0	100	

[1]LSD means least significant digit

During the experiments, we measured the nominal DC voltages x_{nom} = 1, 5, 10, 25, 50 V from the output of Fluke 5520A high-precision calibrator. The nominal value uncertainties were much lower than the uncertainties of the results obtained with multimeters. For example, the uncertainties were ± 0.00008 V and ± 0.00105 V at x_{nom} equal to 10 and 50 V correspondingly. Measurements were made once by each of five digital multimeters. Measurement results x_k and their uncertainties u_k for all nominal values are presented in Table 2.

The data from Table 2 were used to obtain by the IF&PA method an estimate of the reference values x^{**} and their uncertainties u^{**} for different nominal values. For comparison, the same data were processed by the method of weighted mean (WM) and estimates of the reference value y were obtained by the formula [20, 21]:

$$y = \sum_{k=1}^{m} u_k^{-2} x_i \Big/ \sum_{k=1}^{m} u_k^{-2} \quad (8)$$

with corresponding uncertainty:

$$u = \left(1 \Big/ \sum_{k=1}^{m} u_k^{-2}\right)^{1/2},$$

(9)

where $m = 5$ is the number of multimeters.

Absolute deviations ξ^{**} and ξ of the obtained estimates x^{**} and y from the nominal value x_{nom}, obtained by the methods IF&PA and WM appropriately, were calculated as follows:

$$\xi^{**} = |x_{\text{nom}} - x^{**}|, \quad (10)$$

$$\xi = |x_{\text{nom}} - y|. \quad (11)$$

Figure 1 shows the initial intervals and also reference values and their uncertainties obtained by the two methods IF&PA and WM. In Figure 1, the solid red line shows the nominal value x_{nom}, the solid green line – x^{**}, the solid blue line – y. The green and blue dashed lines show the boundaries $x^{**} \pm u^{**}$ and $y \pm u$ correspondingly.

Computation results obtained by methods IF&PA and WM for the data from Table 2 are reduced to Table 3. From the data in Table 3 and Figure 1, it can be seen that the nominal value x_{nom} is close to both values of the estimate x^{**} and y, while the uncertainty values u^{**} are always less than u. The considerable difference between uncertainty values for IF&PA and WM can be clearly indicated. This means that the IF&PA method gives better results than WM.

Table 2. Measurement results x_k and their uncertainties u_k.

Symbol/ x_{nom}	1 V		5 V		10 V		25 V		50 V	
	x_i, V	u_i, V	x_i, V	u_i, V	x_i, V	u_i, V	x_i, V	u_i, V	x_i, V	u_i, V
M1	1.003	0.008	5.00	0.05	10.01	0.08	25.0	0.4	50.1	0.5
M2	0.998	0.007	4.98	0.05	9.98	0.07	24.8	0.4	49.7	0.5
M3	0.9993	0.0013	4.996	0.007	9.993	0.012	24.97	0.05	49.95	0.07
M4	0.9997	0.0006	5.0002	0.0046	10.002	0.007	25.004	0.011	50.008	0.046

M5	1.008	0.01	5.03	0.06	10.07	0.1	25.17	0.2	50.3	0.6

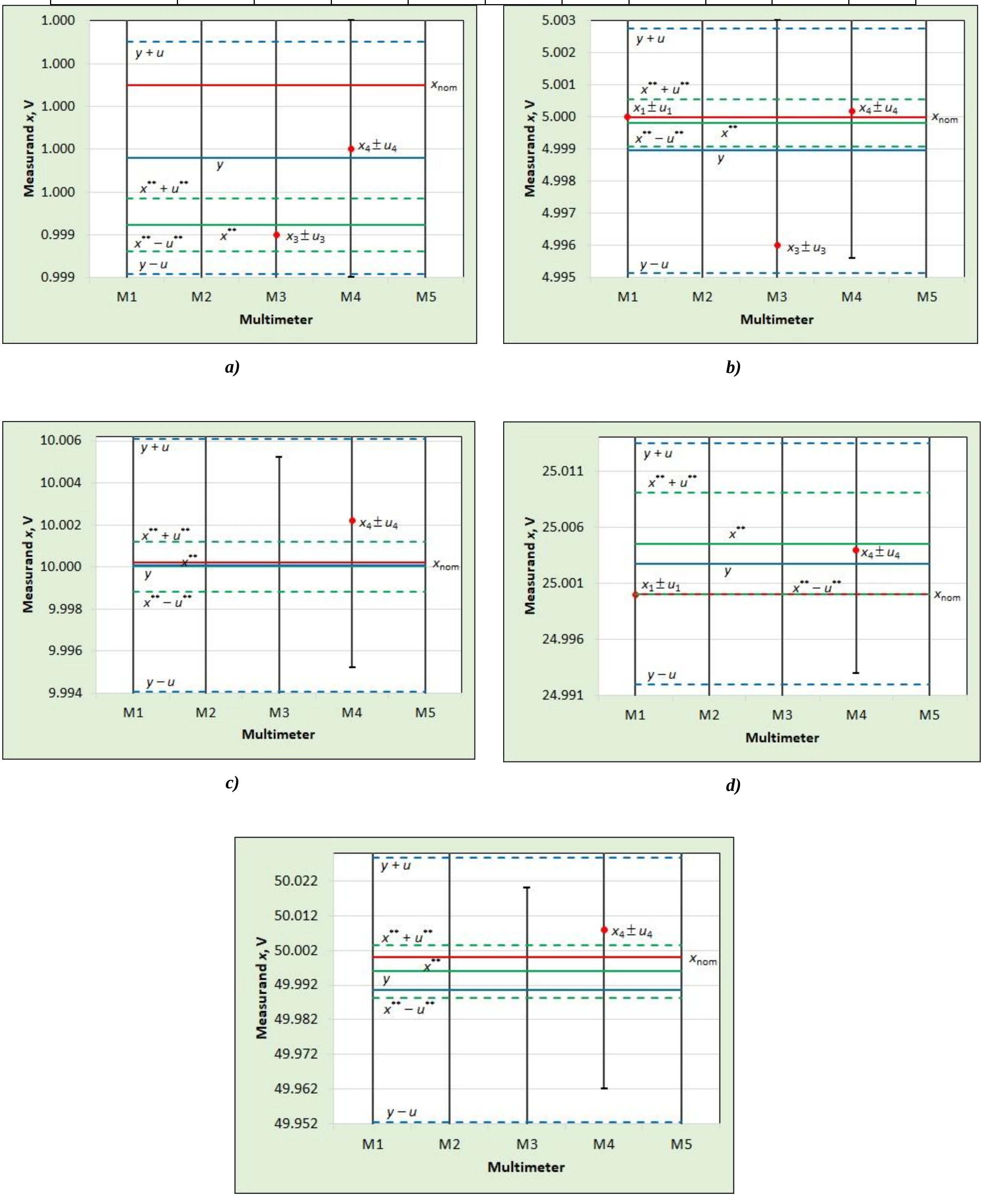
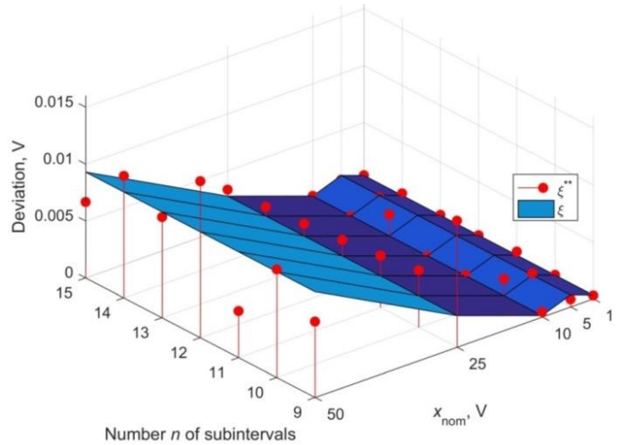


Fig. 1. Estimate x^{**} and its uncertainty u^{**} (green lines), obtained by the IF&PA, and estimate y and its uncertainty u, obtained by the WM (blue lines) at nominal values (red lines): a) 1 V, b) 5 V, c) 10 V, d) 25 V and e) 50 V.

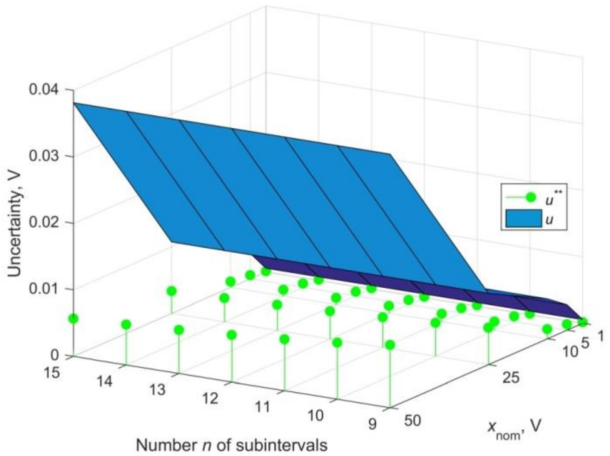
Table 3. Results obtained by the IF&PA and the WM.

x_{nom}, V	IF&PA			WM		
	x^{**}	u^{**}	ξ^{**}	y	u	ξ
1	0.9993	0.0001	0.0007	0.9997	0.0005	0.0003
5	4.9998	0.0007	0.0002	4.9989	0.0038	0.0011
10	9.9998	0.0012	0.0002	9.9999	0.0060	0.0001
25	25.0046	0.0045	0.0044	25.0028	0.0107	0.0028
50	49.9959	0.0077	0.0041	49.9907	0.0381	0.0093

Figure 2 shows the deviations of the reference value from x_{nom} obtained by the methods IF&PA and WM, and the corresponding uncertainties depending on the number of n sub-intervals (see Step 1 in Section 2) and the nominal values. As can be seen from Figure 2a, the deviation value is varied for different values of n. It is clear that, at $n = 11$, the IF&PA demonstrates the advantage in determining the reference value. Figure 2a and data from Table 3 let one notice that, in the majority of cases, the reference values obtained by IF&PA and WM differ only slightly, which can allow to consider the IF&PA estimate as a reliable one. In its turn, Figure 2b shows the great difference in uncertainty of reference values for two investigated methods. The uncertainty values, produced by the IF&PA procedure, are in average tenfold lower than the ones produced by the WM method. From Figure 2b it follows the undeniable superiority of the IF&PA method in terms of the reference value uncertainty.



a)



b)

Fig. 2. Obtained by the methods IF&PA and WM: a) deviations of reference values from x_{nom} and b) corresponding uncertainties – in dependence on number n of subintervals and nominal values.

V. CONCLUSIONS

On real life experimental data, it is shown in the paper that the presented modified IF&PA procedure allows to obtain an accurate reference value based on the heteroscedastic initial interval data. The experimental results demonstrate that, in the majority of cases, the reference values obtained by the modified IF&PA procedure and the traditional WM method are very close to each other, which allows to conclude about the reliability of the IF&PA results. At the same time, the modified IF&PA allows to obtain, for initial heteroscedastic intervals, a fusion result (reference value) with considerably reduced uncertainty in comparison with traditional method based on weighted mean calculation. The obtain results show that the modified IF&PA procedure can be effectively applied for heteroscedastic data processing, in particular when the number of initial measured values is rather small.

VI. ACKNOWLEDGMENTS

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