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Testing Data Converters when Sampling is Incoherent

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Abstract - Coherent sampling is a well known scheme for guaranteeing uniform distribution of sinewave phases. While such condition can be profitably exploited for testing Analog to Digital Converters (ADC) or Digital to Analog Converters (DAC), inaccuracies and non-idealities in both the signal generator and the sampling hardware make such condition rarely achievable. In this paper, the effect of deviations from the coherent sampling condition is analyzed, along with some effects on the related testing procedures.
Keywords – Analog to Digital Converter (ADC) and Digital to Analog Converter (DAC) testing, Coherent Sampling, Farey Series

I. Introduction

Analog to Digital Converter (ADC) and Digital to Analog Converter (DAC) testing is an important activity, subject to several research and standardization efforts [1][2]. Such devices are usually tested by sinewaves as test signals. To properly test a converter, the test design requires that the signal parameters and record length should be properly selected, in order to guarantee that each converter level is excited. As shown in [3][4], some criteria have been proposed, which rely on proper selection of sinewave parameters, especially frequency, amplitude, and record length. In particular, coherent uniform sampling, obtained when the ratio between the sampling frequency and the sinewave frequency is a rational number, results in uniform sinewave phase distribution. Consequently, by suitably choosing such frequency ratio and the record length, it is possible to guarantee the excitation of each converter level [4]. However, coherent sampling cannot usually be obtained in practice, due to the limited accuracy achievable when selecting or measuring both the sinewave and the sampling frequencies. Consequently, some theoretical results have been developed, describing the loss in uniformity in the sinewave phase distribution, when the ratio between sinewave and sampling frequency slightly deviates from the selected value [3][4]. Moreover, due to the recent attention given to the standardization of DAC testing, a deeper analysis of this topic is of significant interest. In this paper, bounds to the achievable accuracy in the selection of the sinewave phases are established, as a function of the achievable accuracy in the frequency selection. Moreover, it is shown that, when the sinewave frequency complies with the reported bounds, a moderate increase in the record length allows the achievement of the desired phase accuracy, even when the sampling is not coherent. The obtained results are used to discuss a potential improvement to the procedure described in [4].

II. Coherent sampling and converter testing

Let us assume that the samples of sinewave record $s[\cdot]$ are given by:

$$s[n] = A\sin(2\pi\rho n + \varphi), \qquad n = 0,..,M-1$$
$$\rho = \frac{f}{f_s} \tag{1}$$

where A is the sinewave amplitude, φ is the initial phase, which in the following will be assumed equal to zero without loss of generality, and ρ is the ratio between the sinewave frequency f and the sampling frequency f_s.
Let us also assume that

$$\rho = \rho_0 = \frac{f_0}{f_s} = \frac{J}{M}, \tag{2}$$

where J and M are two positive and relatively prime integer numbers, that is $J \perp M$, such that $J<M$.
Let us define the phase number $p[\cdot]$ as

$$p[n] = \left(\frac{J}{M}n\right) \bmod M, \qquad n = 0,..,M-1, \tag{3}$$

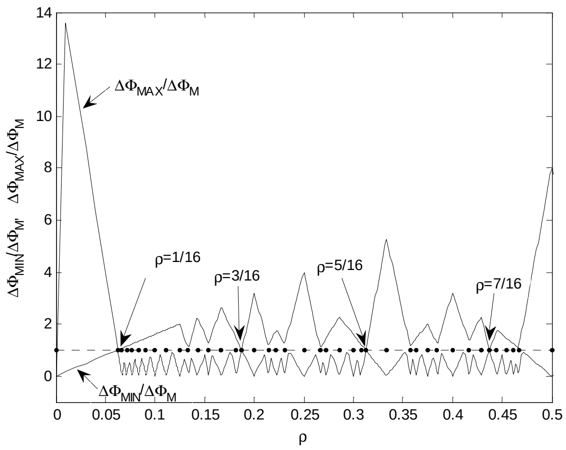

Fig. 1: Maximum and minimum values of the phase gap, reported as a function of the ratio ρ between the sinewave frequency and the sampling frequency, for M=16.

where mod is the modulus operator, [5]

$$x \bmod y = x - y\lfloor x/y \rfloor, \qquad (4)$$

and $\lfloor\cdot\rfloor$ is the floor operator [5]. Consequently, the normalized phase $\phi[\cdot]$ corresponding to each sample is given by

$$\phi[n] = 2\pi p[n]\,, \qquad n = 0,..,M-1 \qquad (5)$$

Under the condition $J \perp M$ the sinewave is coherently sampled, and, provided that the record length is M, it can be shown that M distinct phase numbers are obtained, which are spaced by $1/M$ [3][4]. Thus, the normalized phases are uniformly distributed in [0,2π], and the gap $\Delta\phi$ between adjacent phases is constant and equal to $\Delta\phi_M=2\pi/M$. In order to properly test a converter, the record samples should guarantee that each of the converter bins is excited at least once. By assuming that the device to be tested is a uniform and bipolar converter, with Full Scale FS and b bits of resolution, it has been shown that when the signal is coherently sampled and $FS=A$, the condition

$$\Delta\phi_M \le 2^{-(b-1)} \qquad (6)$$

guarantees that each converter bin is excited at least once by the record samples [4]. Under the introduced hypotheses, such condition is satisfied when

$$M \ge 2\pi\, 2^{(b-1)}\,, \qquad (7)$$

which gives a lower bound on the record length M. When sampling is not coherent, however, the gap $\Delta\phi$ may noticeably differ from $\Delta\phi_M$, and even when the minimum value of M is chosen according to (7), some converter bins may not be properly excited. This phenomenon is discussed in the following section.

III. Effects of non coherent sampling

Let us assume that the frequency ratio ρ slightly deviates from the desired value ρ_0 given by (5) to an effective value ρ_E, that is

$$\begin{aligned} \rho_E &= \rho_0 + \Delta\rho = \frac{f_0+\Delta f}{f_s} = \frac{J+\Delta J}{M} \\ \Delta f &= \frac{\Delta J}{M} f_s\,,\ \Delta\rho = \frac{\Delta J}{M} \end{aligned}\,, \qquad (8)$$

where ΔJ is a real number, $\Delta\rho$ is the variation of the frequency ratio, and Δf is the frequency deviation from the desired value f_0. In this case, it has been shown that, for small values of ΔJ, $\Delta\phi$ can assume two distinct values $\Delta\phi_1$ and $\Delta\phi_2$ [3][4]. Moreover, by using the results in [3] and [4], it can also be shown that the solutions are given by

$$\begin{aligned} \Delta\phi_1 &= \Delta\phi_M\left(1+\Delta J\cdot M_L\right) \\ \Delta\phi_2 &= \Delta\phi_M\left(1-\Delta J\cdot M_R\right) \end{aligned} \qquad (9)$$

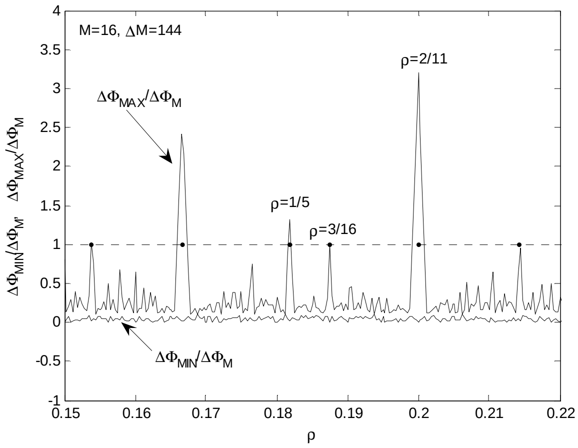

Fig. 2: Maximum and minimum values of the phase gap, reported as a function of the ratio ρ between the sinewave frequency and the sampling frequency, for M=16 and ΔM=144.

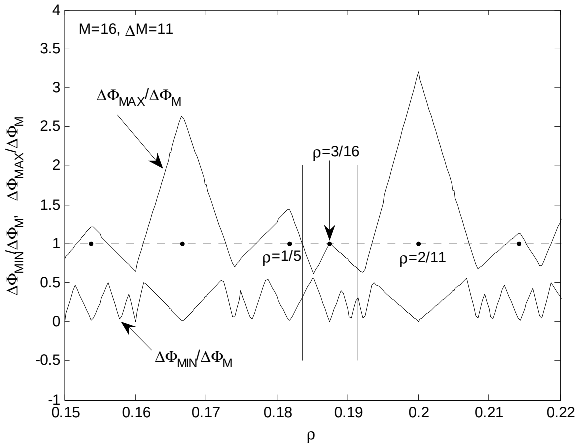

Fig. 3: Maximum and minimum values of the phase gap, reported as a function of the ratio ρ between the sinewave frequency and the sampling frequency, for M=16 and ΔM=11.

where M_L and M_R are the denominators of the left and right fractions adjacent to the fraction J/M in a Farey series (see Appendix A) of order M, such that [5]-[7]

$$\frac{J_L}{M_L} < \frac{J}{M} < \frac{J_R}{M_R}, \qquad 0 < M_L < M, \;\; 0 < M_R < M, \tag{10}$$

Proof of uniqueness of (9) is given in Appendix B. In order to analyze the frequency range in which (9) applies, simulation have been run, leading to the results of Fig. 1, which report the maximum value $\Delta\phi_{MAX}$ assumed by $\Delta\phi$ and the minimum value $\Delta\phi_{MIN}$ assumed by $\Delta\phi$, both normalized to $\Delta\phi_M$, when frequency ratio ρ is varied in the interval [0,1/2] and a record of M=16 samples is generated. Notice that the behavior of $\Delta\phi$ when the frequency ratio ρ is varied in the interval [1/2,1] has been omitted, because it is symmetric with respect to the reported range [0,1/2]. To provide a reference, a dotted horizontal line is also reported. The line height is unitary, and the dots abscissas correspond to fractions belonging to the Farey series of order M. As expected, $\Delta\phi_{MAX}$ and $\Delta\phi_{MIN}$ both coincide with $\Delta\phi_M$ only for a few values of ρ, which correspond to terms of the Farey series with denominator equal to M. Moreover it can be observed that, when ρ is varied from one of such values, $\Delta\phi_{MAX}$ and $\Delta\phi_{MIN}$ deviate linearly from $\Delta\phi_M$ as described by (9), but only when ρ lies in the interval defined by the two adjacent terms of the Farey series, that is

$$\frac{J_L}{M_L} \le \rho \le \frac{J_R}{M_R}, \tag{11a}$$

from which we obtain the following bound on ΔJ (see Appendix B):

$$-\frac{1}{M_L} \le \Delta J \le \frac{1}{M_R} \tag{11b}$$

Moreover, by recalling the definition of the frequency deviation Δf, a bound on the accuracy of the sinewave generator can be derived from (11), given by

$$-\frac{f_s}{MM_L} \le \Delta f \le \frac{f_s}{MM_R} \tag{11c}$$

Outside of this interval it can be shown that the obtained values for $\Delta\phi$ may no longer behave according to (9). In particular, at least one between $\Delta\phi_{MIN}$ and $\Delta\phi_{MAX}$ does no longer satisfy (9). For example, with J/M= 3/16, when ΔJ exceeds the upper bound and the frequency ratio ρ exceeds the right adjacent Farey term, which is 1/5, (9) does no longer provide the correct value of neither $\Delta\phi_{MIN}$ nor $\Delta\phi_{MAX}$, and when the frequency ratio ρ becomes lower than the left adjacent Farey term, which is 2/11, $\Delta\phi_{MIN}$ is no longer given by (9). It is also worthy of notice that, when Δf equals one of the bounds of (11c), the sinewave is actually coherently sampled, but the frequency ratio ρ is an irreducible fraction, with a denominator lower than M. Hence, the gap between adjacent phases is larger than $\Delta\phi_M$, ($2\pi/M_L$ or $2\pi/M_R$), and cannot be reduced by increasing the record length. In the following section, the problem is further discussed.

IV. Selection of record length and sinewave frequency

In [4], it has been suggested that, by increasing the record length M by ΔM samples, the increased phase $\Delta\phi_{MAX}$ can be reduced below the desired value $\Delta\phi_M$. However, it can be observed that, when ρ complies with the bound (11a) (and Δf complies with (11c)) but assumes values close to the reported limits, the phase gap can actually be reduced below $\Delta\phi_M$ only when the record length is largely increased. This can be observed in Fig. 2, which has been obtained under the same conditions of Fig. 1, except for the record length, which has been increased by 10 times, and amounts to 160 samples. It is worthy of notice that such an increment leads to values of $\Delta\phi_{MAX}$ much lower than the required value $\Delta\phi_M$. Moreover, low values of $\Delta\phi_{MAX}$ are obtained only when ρ is not close to the bounds (11a). In fact, in this case the maximum observed frequency gap is still larger than $\Delta\phi_M$. Consequently, a large increase in the record length may not be a desirable solution, because of both the increased time consumption and the possibility of still having undesirable values of $\Delta\phi_{MAX}$. Following the results of section III, the problem has been further analyzed. In particular, it can be shown (see appendix B) that, provided that $\Delta\rho$ and Δf satisfy the equivalent bounds

$$|\Delta f| \le \frac{f_s}{M^2}, \quad (12a)$$

$$|\Delta\rho| \le \frac{1}{M^2}, \quad (12b)$$

the requirement $\Delta\phi_{MAX}<\Delta\phi_M$ can be satisfied by choosing

$$\Delta M=\max(M_L,M_R). \quad (13)$$

Consequently, if (12) is satisfied, by adding at most M-1 samples, the requirement on the phase gap can be satisfied for any initial choice of J/M. For example, by considering the frequency ratio ρ_0=3/16, the denominators of the adjacent terms of the Farey series are M_L=11 and M_R=5 respectively. Hence, by choosing ΔM=11, the results shown in Fig. 3 have been obtained, where the vertical lines mark the upper and lower bounds given by (12). As expected, the values of ρ chosen according to (12) correspond to phase gaps upper bounded by $\Delta\phi_M$. Consequently, provided that the frequency error cannot be accurately controlled and method A in [4] cannot be applied, an alternative approach to test procedure B in [4] can be developed. In fact ρ_0 could be chosen as one fraction of the Farey series with denominator equal to M, but such that both the denominators M_L and M_R of the adjacent terms are approximately equal to $M/2$. While this condition would not maximize the acceptable deviations of $\Delta\rho$ and Δf given by (12), it would minimize the number of samples (13) to be added in order to satisfy the requirement $\Delta\phi_{MAX}<\Delta\phi_M$. Moreover, the denominators of the adjacent terms M_L or M_R determine how much the gap between adjacent phases is enlarged when the frequency ratio ρ equals the bounds (11). In this regard, according to Fig. 1, the best choice for the frequency ratio would be ρ_0=7/16, corresponding to M_L=7 and M_R=9. It can be observed that in this case, the increase in $\Delta\phi_{MAX}$ when ρ approaches the bounds (11a) is minimized with respect to other choices of ρ_0.

V. Conclusions

The loss in phase uniformity associated to non coherent uniform sampling of a sinewave has been investigated, and upper bounds to the inaccuracy associated to both sinewave and sampling frequency have been identified. Under such conditions, it is shown that the requirements on the gap between adjacent phases can be satisfied also when sampling is not coherent, by moderately increasing the record length.

Appendix A: Basics of Farey Series

A Farey series of order M is the set of nonnegative fractions whose numerator and denominator are relatively prime integers, such that the numerator is lower than the denominator, which in turn is lower or equal to M [5]-[9]. By defining the mediant of two fractions J_1/M_1 and J_2/M_2 as $(J_1+J_2)/(M_1+M_2)$, and by defining the Farey series of order 1 as the set of fractions 0/1 and 1/1, a Farey series of order M may be recursively generated from the series of order M-1 [5]. This can be done by evaluating all of the mediants between consecutive elements of the Farey series of order M-1 and by inserting the mediants with denominator not exceeding M between the corresponding terms of the Farey series of order M-1 [5]. In particular, it can be shown that any nonnegative fraction $0<J/M<1$ can be obtained as a term of the Farey series of order M, provided that J and M are relatively prime [5]. Farey series are a powerful tool to analyze coherent sampling of sinewaves, as such a condition implies that the ratio between the sinewave frequency and the sampling frequency is by definition a term of a Farey sequence.

Appendix B: Theoretical results

Existence and uniqueness of solutions (9)

In [3] it is shown that, when the sampling frequency slightly drifts from the coherency condition, a couple of phase numbers is obtained in the form

$$\begin{aligned}\Delta\phi_1 &= \Delta\phi_M\left(1+\Delta J\cdot K\right)\\ \Delta\phi_2 &= \Delta\phi_M\left(1-\Delta J\cdot(M-K)\right),\end{aligned} \tag{B.1}$$

where $0 \le K < M$ is a solution of the equation $KJ \bmod M = 1$. Conversely, in [4] two solutions are obtained as

$$\begin{aligned}\Delta\phi_1 &= \Delta\phi_M\left(1+\Delta J\cdot m_L\right), \quad m_L = M_L \bmod M, \quad 0 < m_L < M\\ \Delta\phi_2 &= \Delta\phi_M\left(1-\Delta J\cdot m_R\right), \quad m_R = M_R \bmod M, \quad 0 < m_R < M\end{aligned} \tag{B.2}$$

As $0<M_L<M$ and $0<M_R<M$, it follows that $m_L=M_L$ and $m_R=M_R$, and the properties of the Farey series imply that $M_L+M_R=M$, because J/M is the mediant of J_L/M_L and J_R/M_R. Moreover, it is easily proven that m_L and m_R are unique solutions of the corresponding equations (B.2). In fact, if for example m_{L1} and m_{L2} were two distinct solutions of $m_L=M_L \bmod M$, then both m_{L1} and m_{L2} would belong to the modulo M equivalence class, and they would differ by nM, where n is an integer number. As by definition both m_{L1} and m_{L2} are positive and upper bounded by M, it follows that the only suitable value of n is 0, that is $m_{L1}=m_{L2}$. Hence, the solutions (B.1) and (B.2) coincide, and are given by the denominators M_L and M_R of the terms of the Farey series of order M adjacent to J/M, leading to (9).

Proof of bounds (11) and (12)

In [4], it is shown that when the numerator J of frequency ratio deviates by an amount ΔJ, then, for $\Delta J>0$, M_R phase gaps are increased by $2\pi\Delta J M_L/M$, while M_L gaps are decreased by $2\pi\Delta J M_R/M$. Conversely, when $\Delta J<0$, M_L gaps are increased by $2\pi|\Delta J|M_R/M$, while M_R gaps are decreased by $2\pi|\Delta J|M_L/M$. As a corollary, regardless of the sign of ΔJ, the worst case enlarged phase gap is given by

$$\Delta\phi_{MAX} = \frac{2\pi}{M}\left(1+\max\left(M_L, M_R\right)|\Delta J|\right) \tag{B.3}$$

Bounds on ΔJ for which (9) holds can be derived by observing that (see Eq. 13 in [4]), if $\Delta J>0$, the first M_R samples of the acquired record correspond to lower extremes of the enlarged phase gap. In particular, if n is the index of any of these samples, the higher extreme of the enlarged phase gap corresponds to the sample with index $n+M_L$. Moreover, for small values of $|\Delta J|$, if the record length is increased by M_R samples, the corresponding phase numbers are adjacent to the ones of the first M_R samples of the record, differing from them by exactly $M\Delta J$. Consequently, each of the M_R enlarged gaps, whose amplitude equals $\frac{2\pi}{M}\left(1+M_L\Delta J\right)$, is broken in two smaller gaps, one with amplitude $A_1=\frac{2\pi}{M}\left(1+M_L\Delta J-M\Delta J\right)=\frac{2\pi}{M}\left(1-M_R\Delta J\right)$ and one with amplitude $A_2=2\pi\Delta J$. Notice that such condition is guaranteed when

$$\Delta J < \frac{1}{M_R}. \tag{B.4}$$

Otherwise, any additional sample with index $n+M$ may produce a phase laying outside the enlarged phase gap corresponding to the sample with index n. When $\Delta J<0$, by following a similar reasoning it can be shown that, if the bound

$$\Delta J > -\frac{1}{M_R} \tag{B.5}$$

is satisfied, the M_L enlarged gaps, whose amplitude equals $\frac{2\pi}{M}\left(1-M_R\Delta J\right)=\frac{2\pi}{M}\left(1+M_R|\Delta J|\right)$, may be broken in couples of gaps with amplitude $B_1=\frac{2\pi}{M}\left(1+M_R|\Delta J|-M|\Delta J|\right)=\frac{2\pi}{M}\left(1-M_L|\Delta J|\right)$ and $B_2=2\pi|\Delta J|$ by collecting M_L additional samples. Notice that (B.4) and (B.5) are equivalent to the bounds (11), and, regardless of the sign of ΔJ, they are satisfied when $|\Delta J| < \min(1/M_L, 1/M_R)$. Moreover, while $A_1$and B_1 are by definition lower than $\Delta\phi_M$, A_2 and B_2 are also guaranteed to be lower than $\Delta\phi_M$ if

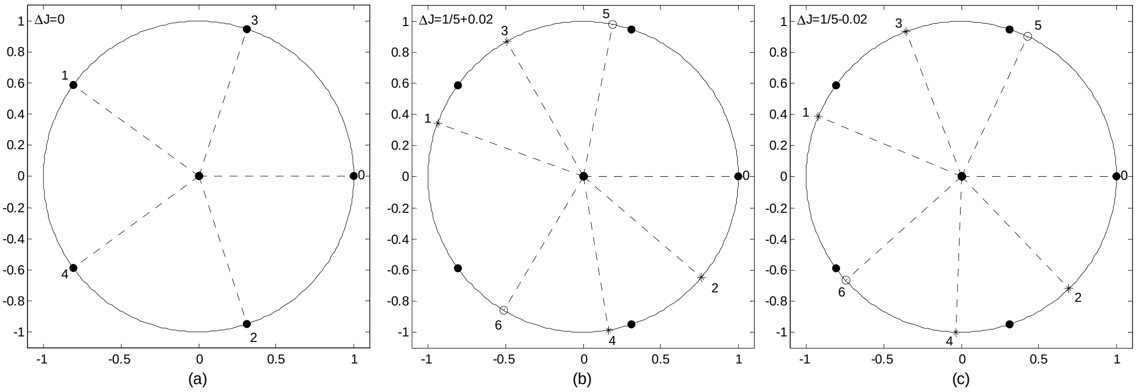

Fig. 4: Phases obtained for J/M=2/5, M_L=3, M_R=2. Dots on the unit circle are ideal phases obtained for a record of M samples when ΔJ =0, asterisks are phases obtained for M samples when ΔJ >0, circlets are phases obtained by collecting M_R additional samples. The numbers next to each phase marker are the corresponding time indexes.

$$\left|\Delta J\right| < \frac{1}{M} \tag{B.6}$$

Notice that (B.6) implies (B.4) and (B.5). Consequently, when the bound (B.6) is satisfied, increasing the record length by max(M_L,M_L) samples ensures that all of the phase gaps are smaller than $\Delta\phi_M$. Finally, by using (8), bounds (12a) and (12b) can be obtained from (B.6). Figure 4a, obtained for J/M=2/5, J_L/M_L=1/3, and J_R/M_R=1/2, shows the phase gaps obtained when a record of M=5 samples is collected and ΔJ=0, while Fig. 4b shows the obtained phase gaps when ΔJ slightly exceeds 1/M, satisfying (B.4) and (B.5) but not (B.6), and M_R=2 additional samples are collected. As expected, the M_R enlarged gaps are partitioned by the new samples, but the amplitude of some reduced phase gaps still exceeds $\Delta\phi_M$=2π/5. Conversely, in Fig. 4c, ΔJ is slightly smaller than 1/M. As expected, after acquiring M_R additional samples, all of the obtained phase gaps are smaller than $\Delta\phi_M$.

References

[1] IEEE Std 1057-1994, “IEEE Standard for digitizing waveform recorders”.
[2] IEEE Std. 1241-2000, “IEEE Standard for terminology and test methods for analog to digital converters”.
[3] P. Carbone, G. Chiorboli, “ADC Sinewave Histogram Testing With Quasi-Coherent Sampling,” *IEEE Trans. on Instrumentation and Measurement*, Vol. 50, No. 4, Aug. 2001.
[4] J. J. Blair, “Selecting Test Frequencies for Sinewave Tests of ADCs,” *IEEE Trans. on Instrumentation and Measurement*, Vol. 54, No. 1, Feb. 2005.
[5] R. L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, 2nd Edition, Addison-Wesley, ISBN 0-201-55802-5.
[6] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, (Fifth Edition), 1979. Oxford University Press, ISBN 0-19-853171-0.
[7] Huxley, M.N. "The distribution of Farey points, I." *Acta Arith.* 18 (1971), 281-287
[8] I. Vaughan L. Clarkson, J. E. Perkins, I. M. Y. Mareels, “Number Theoretic Solutions to Intercept Time Problems,” *IEEE Trans. on Information Theory*, Vol. 42, No. 3, May 1996.
[9] C. E. Pătraşcu, M. Pătraşcu, *Computing Order Statistics in the Farey Sequence*, Springer Berlin / Heidelberg, ISBN 978-3-540-22156-2.