Distribution Laws of Quantization Noise for Sigma-Delta Modulator

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Abstract – The very fast development of sigma-delta analog-to-digital converters is evidently [1, 2]. Therefore analytical methods of their description become more and more important. If the found equations are precise enough, they can be used for verification and validation of modelling and simulation. Analytical equations are also important from the viewpoint of education. As is known, the sigma-delta ADC consists of sigma-delta modulator (SDM) and digital filter. In this paper the quantization noise of modulator is investigated for different forms of input signal. The world known (classical) theory [3-5] of quantization noise is not accurate enough [6]. A new approach was supposed by the authors [7]. The most important point of the approach is to use the discrete two-value distribution law at the modulator output. According to classical theory, for any forms of input signal the quantization noise at the output of the modulator has uniform distribution law. In accordance with new approach, the distribution law of quantization noise depends significantly on the form of input signal. The simulation results are found in agreement with new theory.

I. Distribution laws of quantization noise in analytical form for SDM with different input signals

A. General approach to definition of the distribution laws

The nominal (ideal) transfer function of SDM with input signal X and output signal Y is $Y = X$. The modulator error is $\Delta = Y–X$. Mean value $M(\Delta) = 0$ for the ideal elements of SDM. Output Y for one bit quantizer (comparator) is represented by one of two possible values: $+V_{REF}$ or $–V_{REF}$. As it is used in literature, all signals are shown as referred to value V_{REF} (it means that Y has two possible values +1 and –1 while X lies between –1 and +1). Then only two levels of error take place: $\Delta_1 = –1–X$ for $Y= –1$ and $\Delta_2 = 1–X$ for $Y= +1$. For quantization noise analysis, all modulator components (switches, integrators, and comparator) are supposed to be ideal. At any random moment, the realization of Δ_1 or Δ_2 is a random event. Probability of the first event is $P_1 = 0.5(1–X)$ while probability of the second event is $P_2= 0.5(1+X)$.

The absolute error of SDM produced by quantization noise is:

$$\Delta = \varphi(X) \tag{1}$$

For any value of X with a probability density function $f_1(X)$, the variable Δ has value Δ_1 with probability P_1 and Δ_2 with probability P_2. Therefore the conditional probability density function of variable Δ at any value of X is a δ - function:

$$f_2(\Delta \mid X) = \delta(\Delta - \varphi(X)) \tag{2}$$

The probability density function of two random quantities, X and $\Delta = \varphi(X)$, is:

$$f(X,\Delta) = f_1(X)\delta(\Delta - \varphi(X)) \tag{3}$$

Then the probability density function of Δ can be found as:

$$f_2(\Delta) = \int_{-\infty}^{\infty} f_1(X)\delta(\Delta - \varphi(X))dX \tag{4}$$

B. Distribution law of quantization noise for uniform distribution of input signal

If the input signal X is distributed uniformly from a to b, then $f_1(X) = 1/(b-a)$ and, according to (4), $f_2(\Delta) = P(\Delta)/(b-a)$. According to equations from I.A, probability $P_1 = 1+0.5\Delta$ for SDM output equal to –1 and $P_2 = 1–0.5\Delta$ for SDM output +1. Then probability density function of Δ is described by equations:

$$f_2(\Delta) = \begin{cases} \dfrac{1+0.5\Delta}{b-a}, & \text{for } -1-b \le \Delta \le -1-a \\ 0, & \text{for } -\infty<\Delta<-1-b,\ -1-a<\Delta<1-b,\ 1-a<\Delta<+\infty \\ \dfrac{1-0.5\Delta}{b-a}, & \text{for } 1-b<\Delta\le 1-a \end{cases} \tag{5}$$

The equations (5) can be written as:

$$f_2(\Delta) = \begin{cases} \dfrac{1-0.5|\Delta|}{b-a}, \text{ for } -1-b \le \Delta \le -1-a \text{ and } 1-b < \Delta \le 1-a: \\ 0, \text{ for } -\infty < \Delta < -1-b,\ -1-a < \Delta < 1-b,\ 1-a < \Delta < +\infty \end{cases} \quad (6)$$

The standard deviation can be calculated from (6) as:

$$\sigma(\Delta)^2 = 1 - \frac{a^2 + ab + b^2}{3} \quad (7)$$

Let's use the classical theory [3-5] to describe the modulator output noise. According to this theory, the comparator generates the noise with spectral density:

$$S_{COMP} = 1/3 f_S\ , \quad (8)$$

where f_S – sampling frequency of comparator.

Now let's transform comparator noise to output of the modulator. In accordance with Equation (10) from [7], the variance of this signal in frequency range from zero to any frequency f is:

$$\sigma^2(f) = S_{COMP}\, 2f\left(1 - \frac{\eta}{\pi\tau 2f} arctg \frac{\pi\tau 2f}{\eta}\right), \quad (9)$$

where τ – time constant of the integrator in the first order modulator, η – equivalent coefficient of the comparator in linear model of the modulator. From (8) and (9) the variance at the modulator output is:

$$\sigma^2(f) = \frac{2f}{3f_S}\left(1 - \frac{\eta}{\pi\tau 2f} arctg \frac{\pi\tau 2f}{\eta}\right). \quad (10)$$

The value of η is supposed to equal τf_S for classical theory [3-5] and $2\tau f_S$ for new theory [7]. If $\eta = \tau f_S$ then time delay between output and input of the modulator for sinusoidal signal equals to $1/f_S$. If $\eta = 2\tau f_S$ then time delay between output and input of the modulator for sinusoidal signal equals to $0.5/f_S$. The first value corresponds to maximum delay; the second value corresponds to average delay. The value $0.5/f_S$ is preferable if noise evaluation is carried out. The variance of noise at the frequency range from 0 to $0.5f_S$ for the classical approach is found from (10) as:

$$\sigma^2 = \frac{1}{3}\left(1 - \frac{1}{\pi} acrtg \pi\right), \quad (11)$$

or $\sigma \approx 0.447$. It means that, according to classical theory [3-5], for uniform distributed input signal, output noise of the modulator is also uniform distributed in the range about $\pm 0.447\sqrt{3} \approx \pm 0.773$ with constant probability density function $f_2(\Delta) \approx 0.647$.

Illustration of (5) for $a = -1$, $b = +1$ is given in Figure 1. For these conditions the quantization error at the output of SDM is distributed by Simpson law with standard deviation $\sigma(\Delta) = \sqrt{2}/\sqrt{3}$:

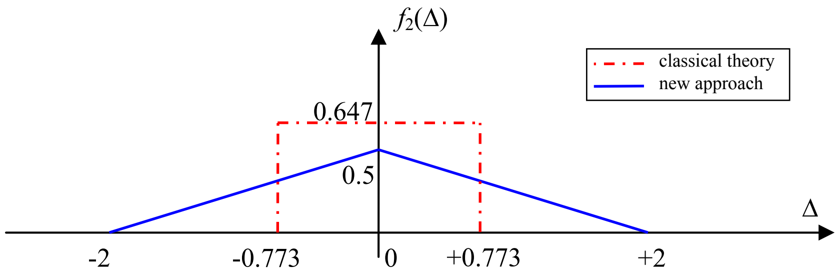


Figure 1. Probability density function of quantization noise for uniformly distributed input signal within [-1; +1]

C. Distribution law of quantization noise for sinusoidal input signal

Let input signal be $X = a\ sin\Omega$, where amplitude a can be chosen from 0 to 1. If error of modulator is considered at random moment of time, then Ω becomes a random quantity uniformly distributed from 0 to 2π. The probability density function of Ω quantity is $1/2\pi$ in the range from 0 to 2π. If we need to find the probability density function $f_1(X)$, the first of all, we should find the cumulative distribution function of X: $F_1(X)$. If $X < -a$ then $F_1(X) = 0$; if $-a \le X \le a$ then $F_1(X) = 0.5 + \dfrac{\arcsin(X/a)}{\pi}$; if $X > a$ then $F_1(X) = 1$. The probability density function of X can be found as a derivative from cumulative distribution function. Then $f_1(X) = 0$ for $X < -a$ or $X > a$ and for $-a \le X \le a$:

$$f_1(X) = \frac{1}{\pi\sqrt{a^2 - X^2}} \quad (12)$$

Using (4) and $X = -1-\Delta_1$ and $P_1 = 1+0.5\Delta_1$ for negative Δ, $X = 1-\Delta_2$ and $P_2 = 1-0.5\Delta_2$ for positive Δ, one can find $f_2(\Delta) = 0$ for $\Delta<-1-a$, $\Delta>1+a$ and $-1+a<\Delta<1-a$. Within interval $1-a \le |\Delta| \le 1+a$:

$$f_2(\Delta) = \frac{1-0.5|\Delta|}{\pi\sqrt{a^2-(1-|\Delta|)^2}} \tag{13}$$

Illustration of (12) and (13) for $a = 1$ is given in Figure 2. The standard deviation can be found from (13) as:

$$\sigma^2 = 1 - 0.5a^2 \tag{14}$$

The maximum value of (14) can be as much as 2.2 times more in comparison with results of classical theory.

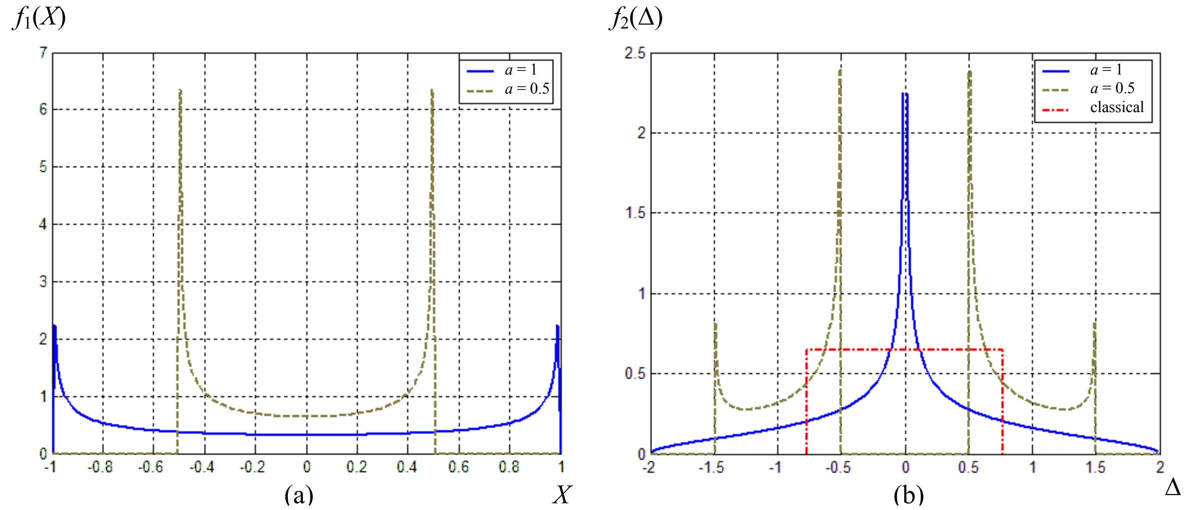


Figure 2. (a) Probability density function for sine input signal with different amplitudes a; (b) Probability density function of quantization noise at the output of SDM for sine input

D. Distribution law of quantization noise for normally distributed input signal (Gaussian)

To keep SDM in linear range $-1 \le X \le 1$, input normal distribution signal is supposed to be truncated by range from $M_X - a$ to $M_X + a$ with following probability density function:

$$f_1(X) = \frac{\lambda}{\sigma_X\sqrt{2\pi}} \cdot e^{-\frac{(X-M_X)^2}{2\sigma_X^2}}, \tag{15}$$

where λ – coefficient which depends on the truncated range a ($1.003 < \lambda < 1.465$ for $\sigma_X < a < 3\sigma_X$),
M_X – mean value, which must correspond to the condition $|M_X| \le 1 - a$,
σ_X – standard deviation.

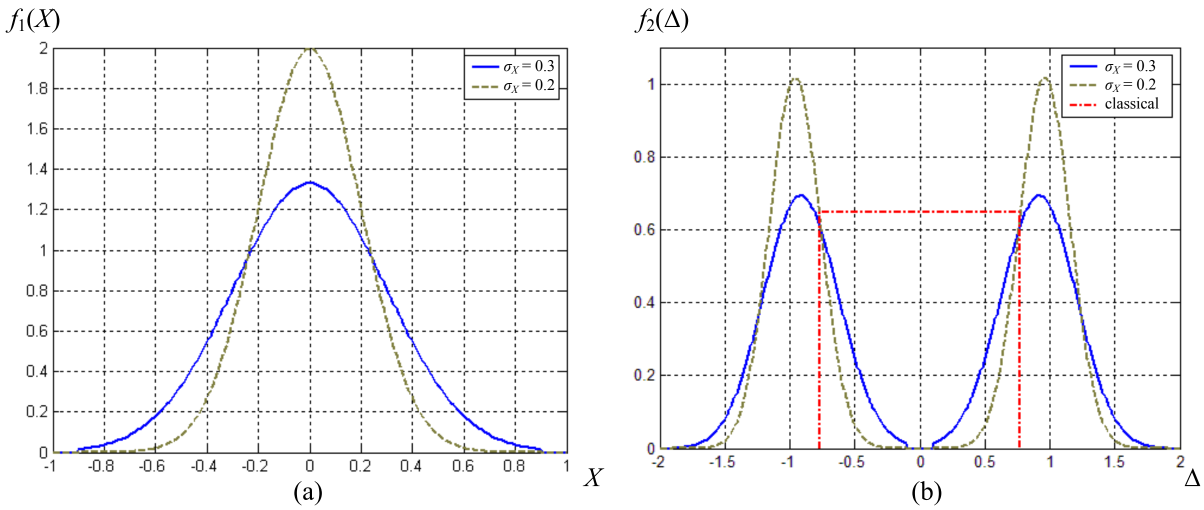


Figure 3. Probability density functions: (a) truncated normal distribution at the input of SDM; (b) corresponding probability density function of quantization error at the output of SDM

As described in I.B: $\Delta_I = -1-X$ with probability $P_1 = 1+0.5\Delta_1$ for negative Δ and $\Delta_2 = 1-X$ with probability $P_2 = 1-0.5\Delta_2$ for positive Δ. If $-a \le X \le a$ then $-1-a \le \Delta_1 \le -1+a$ with $P_1 = 1+0.5\Delta_1$ and $1-a \le \Delta_2 \le 1+a$ with $P_2 = 1-$

$0.5\Delta_2$. Equation for $f_2(\Delta)$ can be found by using (4). Then function $f_2(\Delta) = 0$ at $-\infty \le \Delta \le -1-a$ and $-1+a \le \Delta \le 1-a$ and $1+a \le \Delta \le +\infty$. Within $1-a \le |\Delta| \le 1+a$ the probability density function $f_2(\Delta)$:

$$f_2(\Delta) = \frac{\lambda(1-0.5|\Delta|)}{\sigma_X\sqrt{2\pi}} \cdot e^{-\frac{(-|\Delta|+1-M_X)^2}{2\sigma_X^2}} \qquad (16)$$

For example, if $M_X = 0$, $\sigma_X = \{0.2; 0.3\}$, $\lambda = 1$, $a = 0.9$ then the probability density function calculated by (15) is shown in Figure 3.a. The distribution law at the output of SDM is described by (16) and shown in Figure 3.b.

As you can see in Figure 3.b, the true form of probability density function deflects very much from uniform distribution law predicted by classical theory.

II. Simulation by Matlab

A. Uniformly distributed input signal

We used the application program Matlab 6.5 and built-in simulation program Simulink 5. The scheme is described more detailed in [7].

Uniformly distributed signal within [-1; +1] was applied to the input of SDM. Absolute error between practical results and (6) which is supposed to be true was calculated for different number of input points (length).

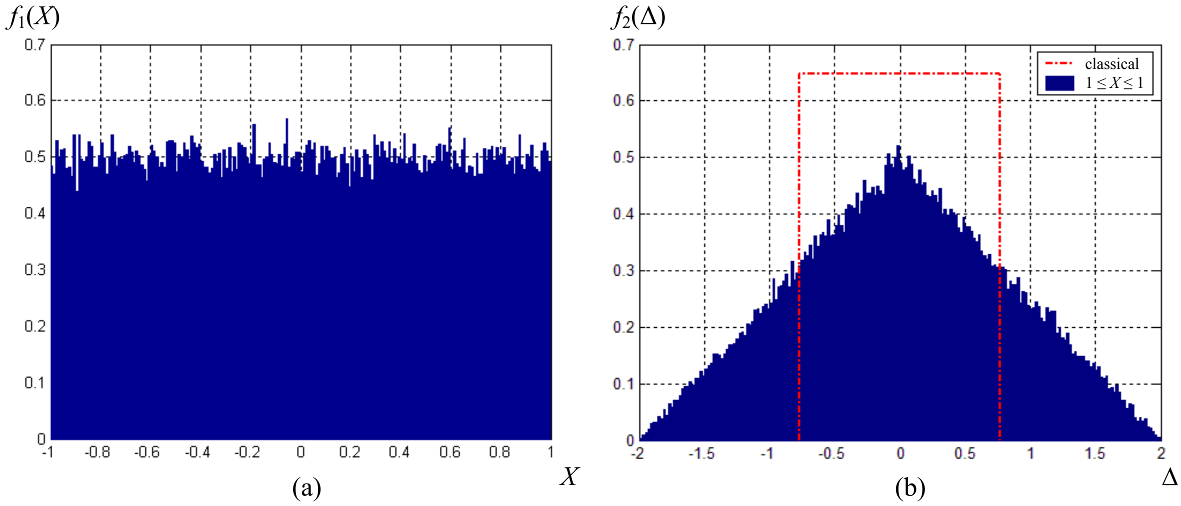


Figure 4. (a) Probability density function of uniform input; (b) Probability density function of quantization error at the output of SDM for uniform input

Histogram of distribution law $f_1(X)$ for $N = 10^5$ points of input signal is shown in Figure 4.a. The histogram has 200 numbers of bins. The corresponding histogram of probability density function $f_2(\Delta)$ for the output of SDM with 200 numbers of bins is shown in Figure 4.b. Absolute error Δ_Δ between simulation results (Fig. 4.b) and (6) (Fig. 1) as function of Δ is shown in Figure 5. As you can see, the absolute error does not exceed 0.041. The standard deviation of absolute error σ_Δ is calculated by:

$$\sigma_\Delta = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\Delta_\Delta - M_{\Delta_\Delta}\right)^2}\ , \qquad (17)$$

where n – number of bins,

M_{Δ_Δ} – mean value of absolute error Δ_Δ.

Value σ_Δ is the index of proximity between simulation histogram of distribution law (Fig. 4.b) and analytical equation (6). We also calculated standard deviation at the output of SDM ($\sigma_{simulation}$) and compared $\sigma_{simulation}$ with (7). Results are shown in Table 1 as dependence of number of input points N.

N	10	10^2	10^3	10^4	10^5
σ_Δ	0.985	0.329	0.106	0.035	0.0118
σ by (7)	$\sigma = \sqrt{2}/\sqrt{3} = 0.817$				
$\sigma_{simulation}$	0.467	0.775	0.832	0.820	0.818
Relative error, %	-74.7	-5.39	1.90	0.458	0.234

Table 1. Comparison of simulation and analytical results as function of point's number for uniform input within [-1; +1]

As is oblivious from Table 1, the difference between histogram of probability density function (Fig. 4.b) and (6) is steadily decreasing by increasing the number of points and can be made as small as it needs. The simulation standard deviation of quantization error approximates to analytical (7) by increasing the number of points N.

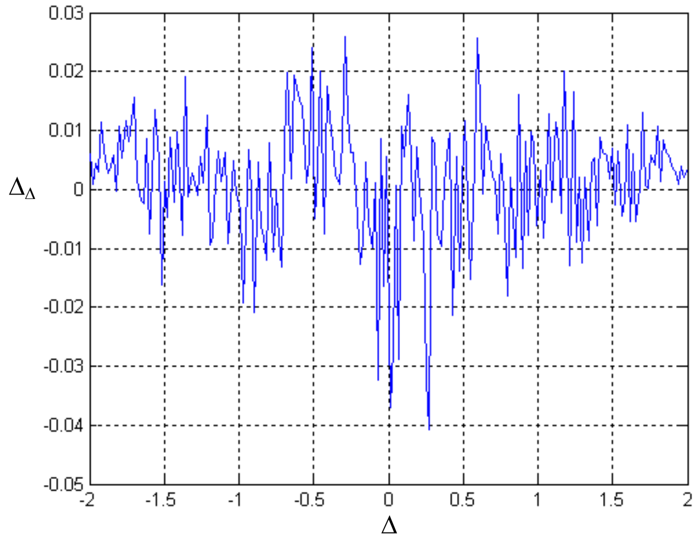


Figure 5. Absolute error between theoretical equation (6) and simulation results

B. Sine input signal

This experiment is similar to the previous one. One period of sine is applied to the input of SDM. The result of simulation is shown in Figure 6 for the number of input points $N = 10^5$.

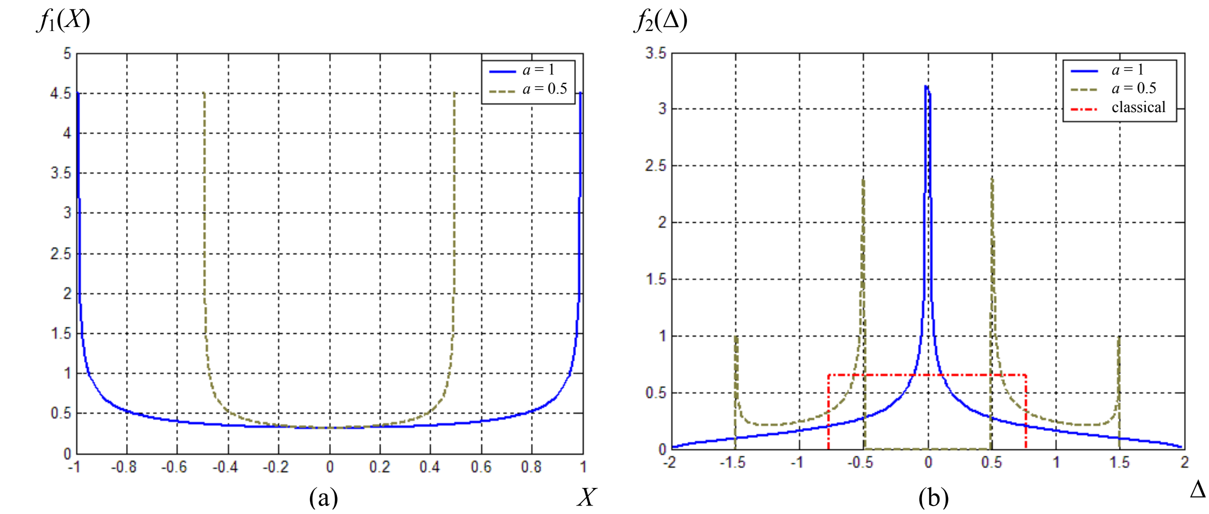


Figure 6. (a) Probability density function of the sine input; (b) Probability density function of quantization error at the output of SDM for the sine input

N	10	10^2	10^3	10^4	10^5
σ_Δ	1.28	0.327	0.119	0.0935	0.0932
σ by (14)	$\sigma = 1/\sqrt{2} = 0.707$				
$\sigma_{simulation}$	0.748	0.708	0.708	0.707	0.707
Relative error, %	5.51	0.130	0.0624	$5.50*10^{-3}$	$5.19*10^{-4}$

Table 2. Comparison of simulation and analytical results as function of point's number for sine input with $a = 1$

In this case the difference between histogram of distribution law (Fig. 6.b) and analytical (13) is steadily decreasing (Table 2) and can be made as small as it needs by increasing the number of points. The simulation standard deviation of quantization error approximates to analytical (14) by increasing number of points N.

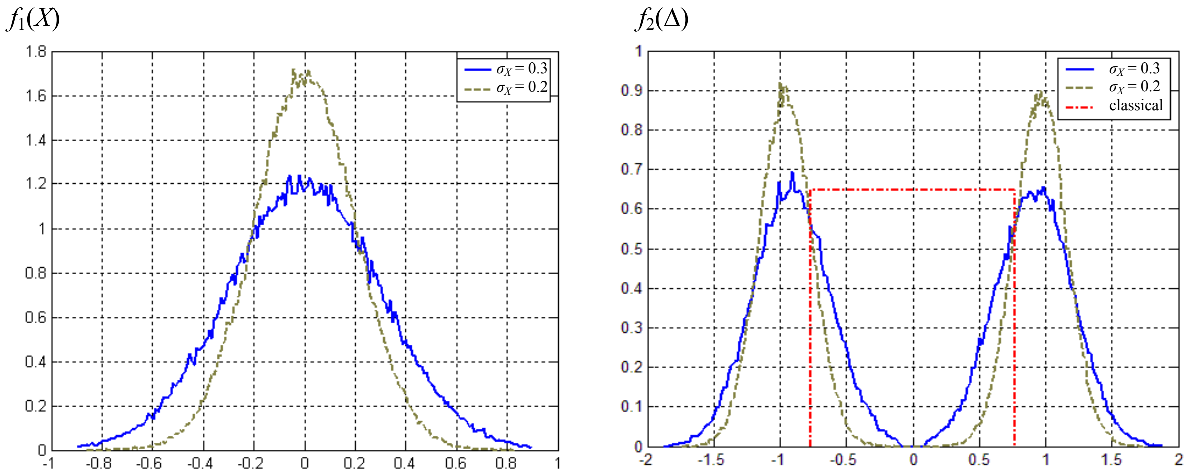
D. Normally distributed input signal (Gaussian)

This experiment is similar to the previous both. Normally distributed signal with mean value $M_X = 0$ and different standard deviations $\sigma_X = \{0.2;\ 0.3\}$ is applied at the input of SDM. Truncated range is $a = 0.9$. The simulation results are shown in Figure 7 for the number of input points $N = 10^5$.

N	10	10^2	10^3	10^4	10^5
σ_Δ	1.02	0.399	0.169	0.0849	0.0504
$\sigma_{simulation}$	1.06	0.957	0.942	0.955	0.955

Table 3. Simulation results as function of point's number for normally distributed input with $\sigma = 0.3$

As it was in previous two cases (uniform and sine), the difference between histogram of distribution law (Fig. 7.b) and analytical (16) is steadily decreasing by increasing number of points N. Results are shown in Table 3.



(a) (b)

Figure 7. Simulation histograms for: (a) truncated normal distribution law at the input of SDM; (b) corresponding probability density function of quantization error at the output of SDM

The results of simulation with described three distribution laws mainly agreeing with simulation results of Hungarian scientist I. Kollar [8].

III. Conclusions

The method of analytical calculation of distribution law at the output of SDM was suggested. The real SDM output distribution law of quantization noise depends significantly on form of input signal and deflects dramatically from uniform distribution law used in classical theory. The true value of standard deviation can be as much as by 2.2 times more as predicted by classical theory.

The simulation results approximate to analytical as close as it needs by increasing the numbers of calculated points. It means that our analytical results are true for chosen structure of SDM.

The new theory can be considered as a reference for validation of SDM simulation.

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