Maximum Likelihood Estimation of ADC Parameters from Sine Wave Test Data

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Abstract **–** The sine wave test is maybe the most important method for characterizing ADC's. By this, the acquisition device is excited with a sinusoidal signal, and a long series of output values is measured. With the help of these observations, the parameters of the DUT can be determined. The general method to do this is the Least Squares (LS). In this paper, we present a similar method using the Maximum Likelihood Estimation (MLE). It is more robust than the LS method, which has nice properties only under special conditions.
This maximum likelihood problem is solvable only numerically. For this, a numerical method is presented, and simulation results are given.
The main message of this paper is how to handle the problems of the estimation in the best way in order to extract possibly the full information from the measured data, and obtain a robust, effective algorithm.
Keywords - IEEE standard 1241-2000, ADC test, analog-to-digital converter, maximum likelihood estimation, Least Squares fit, sine wave fitting, effective number of bits, ENOB.

I. Introduction

Development of digital computers and signal processors is surprisingly quick, and does not seem to slow down. Due to this, companies are developing newer and newer ADCs with better resolution, precision and speed. Hence fast, efficient, and standardized test methods are essential both for producers and for users. Moreover, such test methods reduce the time needed for the buyers of ADCs to validate the performance of the circuits they bought. This is what Standard IEEE-1241 is made for.

Suppose that we have a 12-bit A/D converter. It has 2^{12} output codes and between them 2^{12}-1 = 4095 comparison (or code transition) levels. One of the purposes of ADC testing is to determine these levels (and/or yield simple descriptions of the global error). Once code-transition levels have been measured, then all static parameters, including integral and differential nonlinearities, missing codes, gain, and offset can be computed [1].

Quantitatively, a code transition is the value of the converter-input signal which causes half of the digital output codes to be greater than or equal to, and half less than, a given output code. There exist some procedures to directly measure these, but they are either time-consuming (up to about 1 hour for all levels of a 12-bit ADC) or they are prone to dynamic errors.

In this paper a prospective statistical test method to determine all parameters of the ADC and of the input signal is analyzed, using a single sine wave as excitation. If the sine wave is of low frequency, approximations to the static parameters are determined.

II. Maximum likelihood problem

The general problem statement is as follows. One applies a sine wave as excitation signal. A long series of output values is measured. The input is not accessible. Therefore, the output sequence must be used to determine the most probable parameter values of the input, and then for the extraction of the ADC errors.

Determination of the input signal parameters is usually executed via least squares fit [1]. The method fails when the tacitly applied assumptions (independent, symmetrically distributed error) are violated. This is the case when the ADC is slightly overloaded (this can easily happen when all comparison levels are to be tested, thus one applies a possibly full-scale sine wave), or when the amplitude of the sine wave is only a few times (let us say, <30 times) larger than the ADC quantum

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size. In such cases, modifications (elimination of certain samples) can be used [4], [5]. When using the maximum likelihood estimation, these modifications are not necessary!

Many times, the experimenter is also interested in the noise corrupting the samples. It would be nice to calculate this using the digital samples only. The observation is that noise causes oscillations between the neighboring output levels while the instantaneous value of the input sine is close to these. Thus, a measure of the length of these oscillations can be used to determine the noise level, and the centers of the oscillations can be used to determine the comparison levels. The paper analyses this and gives the estimates. The model is that the input is a sine wave, corrupted by Gaussian white noise, and this is quantized by an ADC. First the ADC is handled as ideal, but this is not a requirement. A typical output signal is illustrated in Figure 1.

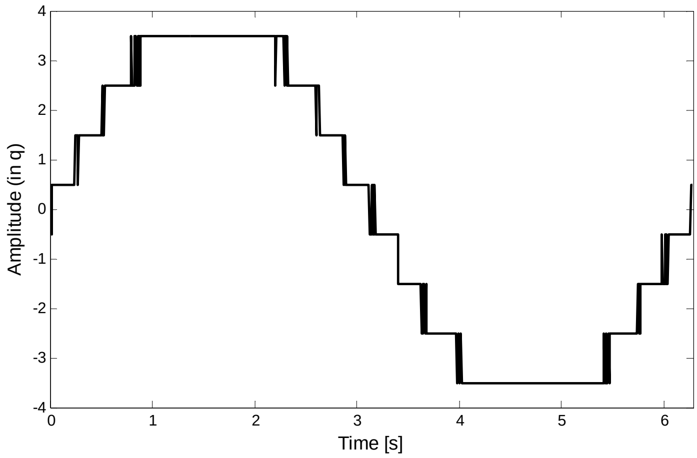

Figure 1. Output samples of an ADC

To obtain initial values, the length is measured at each level of the oscillations, and a weighted sum is formed. The result is corrected on the basis of a nonlinear corrections table [8].

Next, numerical solution of the maximization problem is suggested. Generally, the solution of the maximum likelihood problem is the maximization of the likelihood function $L(\mathbf{a})$ to estimate the unknown parameters ($\mathbf{a}$):

$$\max_{\mathbf{a}} L(\mathbf{a}) = \max_{\mathbf{a}} \prod_{i=1}^{M} P(Y_i|\mathbf{a}) \tag{1}$$

As we will see later, in our case, the likelihood function is a product of probabilities. The distribution of the process is normal. Therefore, this problem is only solvable numerically. If the maximization is done, we can get the best parameters, because of the nice properties of the MLE (consistent, including asymptotically minimum-variance).

The main message of this paper is how to handle this problem in the best way in order to extract possibly the full information from the measured data, and obtain a robust, effective algorithm. The presented method is quasi-static, we suppose, that the excitation signal has low frequency, and under the test conditions no dynamical effects appear (e.g out-of-range recovery).

III. Modeling

The first step of estimation is modeling. We suppose that the ADC is monotonic, and has no missing output codes. Local nonlinearities are possible, the true position of the transition levels have to be estimated. The ideal transfer function of an ADC can be seen in Figure 2.

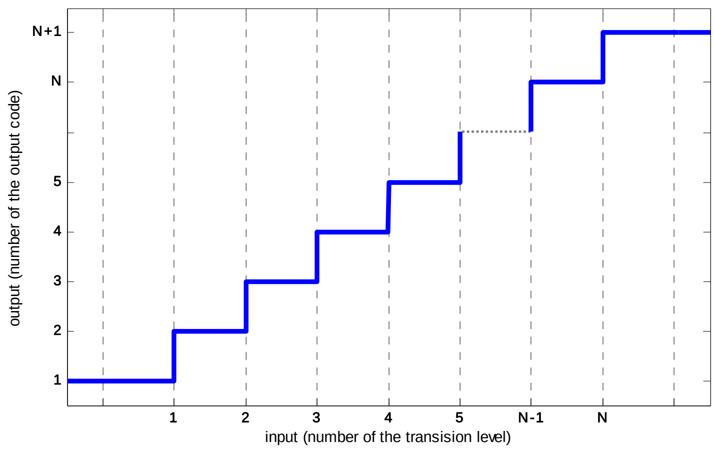

Figure 2. Transfer function of an ideal ADC (N is the number of the transition levels, $N = 2^B - 1$, where B is the number of bits)

When using sine wave test, the exciting signal is sinusoidal:

$$y_n = A\cos(\omega t_n + \varphi) + C = A_0 \cos(\omega t_n) + B_0 \sin(\omega t_n) + C_0 \tag{2}$$

where y_n is the n^{th} output sample, A is the amplitude, ω is the frequency, C is the DC component, A_0, B_0, C_0 are the parameters of the exciting sine wave. The signal at the ADC input is corrupted by some noise. The assumed noise is additive, white, independent, zero-mean and Gaussian. One can say that the sine is the "drift" component of the noise. Therefore, the probability density function (pdf) of the noise is "drifted" in front of the transition levels of the ADC. This can be seen in Figure 3.

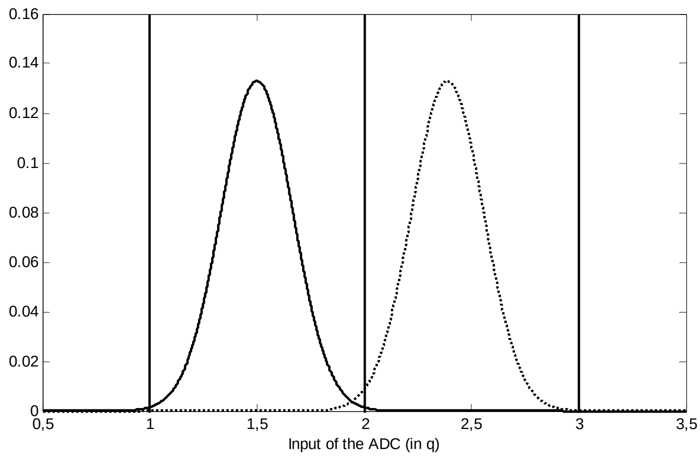

Figure 3. Transition levels and the pdf of the input signal (vertical lines: transition levels, solid curve: pdf of the input signal, dotted curve: "drifted" pdf of the input signal, $\sigma < q$, where σ is the standard deviation of the additive noise, q is the quantum size)

By the maximum likelihood estimation we should maximize the following joint probability:

$$\max_{\mathbf{a}} L(\mathbf{a}) = \max_{A_0, B_0, C_0, \sigma, \mathbf{q}} \prod_{i=1}^{M} P(Y_i = y_i) \tag{3}$$

where L is the so-called likelihood function, **a** is the vector of the parameters, M is the number of the observations, Y_i is the i^{th} observed sample (random variable), y_i is one of the output values (codes), σ is the standard deviation of the additive noise, **q** is the vector of the transition levels.

The probability that the output code is the k^{th} (Y_i = k) is the integral of the pdf between the comparison levels around the k^{th} output code (see Figure 3).

The meaning of (3) is following: with changing the model parameters, the product of the probabilities should be maximal. When we change the parameters, the "Gaussian bell" will be drifted and the

position of the transition levels will be changed, so that this assumption will be true.
Each probability for the intermediate bins can be calculated as follows:

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{q_{k-1}}^{q_k} e^{-\frac{(u-y)^2}{2\sigma^2}}\,du \tag{4}$$

where q_k is the k^{th} comparison level, y is the model of the sine wave, σ is the standard deviation of the additive noise.
The integration limits for the "overdriven" bins are $(-\infty,q_1]$ and $[q_N, \infty)$. Even if the acquisition device is overdriven, the amplitude of the excitation sine wave can be calculated with help of these probabilities
Using the definition of the normal distribution function, the expression (4) can be written as:

$$\Phi\left(\frac{q_k - y}{\sigma}\right) - \Phi\left(\frac{q_{k-1} - y}{\sigma}\right) \tag{5}$$

This expression is solvable only numerically. To solve the ML problem, the likelihood function $L(\mathbf{a})$ has to be maximized. Or equivalently, its logarithm $\log L(\mathbf{a})$ can be used. We can solve the maximization problem by minimizing the function $-\log L(\mathbf{a})$. Therefore, (3) becomes:

$$\min_{\mathbf{a}} \sum_{i=1}^{M} -\log P(Y_i = y_i) \tag{6}$$

IV. Numerical method

The numerical minimization of (3) is a descent method. This performs repeatedly the line search to get a minimum in the direction of the negative gradient. In this paper the so-called backtracking algorithm [9] is applied. Assuming that the current parameter vector is $\mathbf{a}_k$ then $\mathbf{a}_{k+1}$ is computed by

$$\mathbf{a}_{k+1} = \mathbf{a}_k + t \cdot \Delta\mathbf{a}_k \tag{7}$$

where t determines the step size, and the direction of the minimum can be calculated as follows:

$$\Delta\mathbf{a}_k = \left.\frac{\partial -\log L(\mathbf{a})}{\partial \mathbf{a}}\right|_{\mathbf{a}=\mathbf{a}_k} \tag{8}$$

The method of the backtracking line search is the following [9]:
given a descent direction $\Delta\mathbf{a}$ for $-\log L(\mathbf{a})$, $\alpha \in (0, 0.5)$, $\beta \in (0, 1)$:

$t := 1$
while $-\log L(\mathbf{a} + t\Delta\mathbf{a}) > -\log L(\mathbf{a}) - \alpha t \Delta\mathbf{a}^T \Delta\mathbf{a}$, $t := \beta t$

The overall description of the algorithm that calls the gradient method as a subfunction is

1. Calculate initial value for **a**
2. while $err(\mathbf{a}_k, \mathbf{a}_{k-1}) > \varepsilon$
   2.1. Gradient method for only A, B, and C
   2.2. Gradient method for only **q**

where $err(\mathbf{a}_k, \mathbf{a}_{k-1})$ is the absolute error, i.e.

$$err(\mathbf{a}_k, \mathbf{a}_{k-1}) = \left\|\mathbf{a}_k - \mathbf{a}_{k-1}\right\| \tag{9}$$

and ε is a threshold for this error.
A possible solution to determine the initial parameters is using sine fitting, noise estimation algorithm [8] and equal distribution of the transition levels.
The sine wave fitting [1] is a good algorithm for determining the initial signal parameters. At the beginning of the test, we don't have any information about the nonlinearities of the ADC. Therefore, the best we can do is the equally partitioning of the transition levels. To determine the initial value of the additive noise, the earlier presented algorithm [8] is used. This method uses the fact, that the additive noise causes oscillations in the observations. The lengths of these oscillations are proportional to the standard deviation of the additive noise. According to the algorithm, the lengths of the

oscillations have to be measured. An auxiliary-curve is given, with help of its inverse and the measured oscillation lengths the standard deviation of the noise can be determined. The length of the oscillations is searched as a standard deviation, because of their stochastic behavior. The central point of the deviation is determined as the "central of mass". In this case the "mass" is the probability of the oscillation between two output codes. Therefore, the center of the deviation is somewhere in the middle of the oscillation, where the oscillating probability is maximal. It is very similar what the maximum likelihood method does: the method searches the set of the parameters so that the likelihood function (joint probability) is maximal.

V. Simulation

The estimation is realized in MATLAB. The true and the estimated transfer function can be seen in Figure 4. The true positions of the comparison levels are randomly generated, so that the ADC remains monotone. The maximum of the relative error is about 12%.

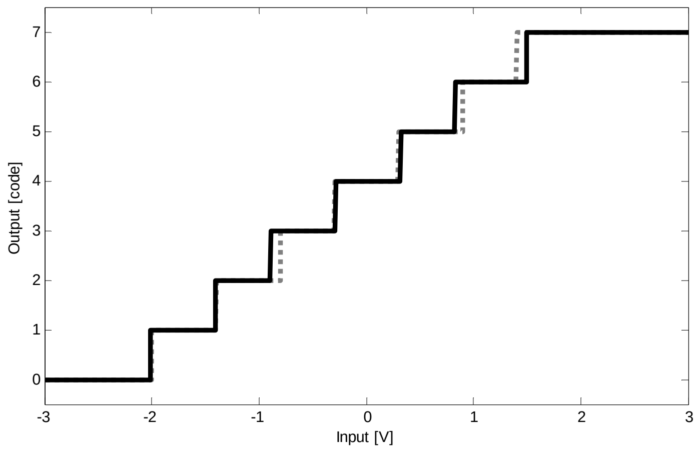

Figure 4. The true (dotted line) and the estimated (solid line) transfer function of the ADC (B = 3bits, the positions of the comparison levels are random, so that the ADC remains monotone)

The noisy input signal and the estimated sine wave can be seen in Figure 5. One can see that the estimation works very well. The true amplitude of the sine is 2, the estimated value is 1.9279.

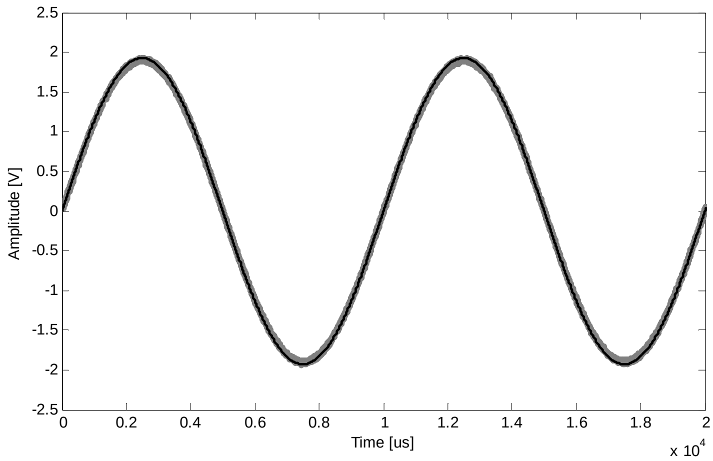

Figure 5. The noisy input signal (grey curve) and the estimated sine wave (black curve) (the standard deviation of the noise is 0.01V)

The estimated value of the standard deviation is 0.05, and the standard deviation of the additive noise is 0.01. Because of the difference between the two values, some correction is necessary. We will do this in the future.

VI. Conclusions

The maximum likelihood estimation has very nice properties. We used this method, and we can say that the sine wave test of the ADC's can be more robust with its help. An important question is the effectiveness. After some calculations we can say, that the Cramer-Rao bound is reached. Details will be published in the future.

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