

Benefits with Truncated Gaussian Noise in ADC Histogram Tests

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Abstract – One method to characterize ADCs is to use a histogram, where Gaussian noise may be used as stimulus signal. However, a Gaussian noise signal that excites all transition levels also generates input values outside working range of the ADC. Modern signal generators can generate arbitrary signals. Hence, excluding undesired values outside the ADC full scale can minimize test sequences. Truncating the signal to the working range gives further advantages, which are explored in this paper. The statistical properties is theoretically evaluated and compared. It is shown that accuracy increases for a fixed sample length and that variation over transition levels decrease.

1. Introduction

The test of analog-to-digital converters (ADCs) with statistical analysis is based on the building of a histogram, which gives the number of occurrences of each code at the output of the converter. This histogram is then compared with the probability density function (pdf) of the stimulus signal. Sine waves are commonly used as stimulus signals, due to the easiness with which they can be generated and with their spectral purity. However, there are problems associated with the use of a sinusoid, or any other deterministic signal [1]. One is that the sinusoid is periodic. The relation between the signal and acquisition frequencies must be arranged in sampling schemes. Another is the (per definition) narrow bandwidth of a sinusoid.

White noise is a wide band signal. In a single characterization run we can thus relate merit figures of the converter to a frequency band. Other benefits for using Gaussian histogram tests (GHT) are that the noise wave is as easy as or easier to generate than a sine wave. Only the mean value and the variance are relevant for the characterization. A Gaussian distributed measurement disturbance only induces a gain error, which can be circumvented.

Modern signal generators can generate arbitrary signals. We can design a noise signal for our purpose; generate it in the test equipment as a stimulus signal and compare the measured histogram values versus the histogram of the generated signal. In this paper we focus on a noise signal truncated to the working range of the ADC.

The ADC histogram test method is considered as an estimation problem where the task is to estimate an arbitrary transition level T_k based on the ADC output. Further, the ADC is modeled as a static nonlinearity so that the output of the device under test is white for a white stimuli signal.

For a white stimuli signal under a Gaussian assumption, the problem was considered in [4]. The statistical properties of GHT was theoretically evaluated and compared to the corresponding Cramér-Rao Lower Bound (*CRLB*). It was shown that the GHT is asymptotically unbiased and efficient.

In this work, we take this further and evaluate a more suitable signal for practical measurements. Truncated Gaussian noise is limited to only consist of useful values, which are values inside working range of the ADC. Not only will this shorten the test sequence, it will also improve the performance of the test as will be shown. The *CRLB* is derived and compared to formerly presented results.

2. Stimulus Signal Properties

When using noise as stimulus the variance cannot be arbitrary. It must be such as to allow all of the 2^n codes of the n -bit converter to be excited. However, a high noise variance implies the existence of values outside the working range, or full scale (*FS*) of the converter.

Earlier studies of noise as stimulus have practical solutions to this problem. Usually, a limiter is used to protect the ADC from unsuitable signals. A limiter will save the component, but the test sequence will be longer than necessary since in contain useless test samples. An alternative to avoid stimulus signals outside working range is to generate the noise off-line from a pseudo-random sequence [2]. Values

outside the ADC working range are excluded in the sequence. An arbitrary signal generator is then used to realize the stimulus signal, which can be described as a truncated Gaussian noise. It can be shown [3] that this signal is described by (1) below. The pdf of a truncated Gaussian noise is given by

$$f(T_k) = \begin{cases} \frac{1}{c\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(T_k - \mu)^2\right] & |T_k| < FS \\ 0 & |T_k| > FS \end{cases} \quad (1)$$

where

$$c = \int_{-FS}^{FS} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(T_k - \mu)^2\right] dT_k \quad (2)$$

In (1)-(2), the set of T_k contains the transition levels. Further, σ is the standard deviation (σ^2 is the variance) and μ is the mean value of the generated Gaussian noise stimuli $s[n]$. The quantity c in (2) is determined by the requirement that $f(T_k)$ integrates to unity. The distribution function is then described by

$$F\left(\frac{T_k - \mu}{\sigma}\right) = \frac{1}{2} + \frac{1}{2c} \operatorname{erf}\left(\frac{T_k - \mu}{\sqrt{2}\sigma}\right) \quad (3)$$

The statistical efficiency of an unbiased estimator can be evaluated by comparing its variance to the corresponding Cramér-Rao Lower Bound (*CRLB*). The *CRLB* of an unbiased transition level estimator with Gaussian stimulus was derived in [4], that is

$$CRLB(T_k) = \frac{\sigma^2}{N} \frac{\Phi\left(\frac{T_k - \mu}{\sigma}\right) \left[1 - \Phi\left(\frac{T_k - \mu}{\sigma}\right)\right]}{\phi\left(\frac{T_k - \mu}{\sigma}\right)^2} \quad (4)$$

In (4), $\Phi(\cdot)$ and $\phi(\cdot)$ are the normalized probability distribution and the probability density function, respectively, given by (1)-(2) with $c=1$. The scalar N is the number of samples. In [4], it is shown that the Gaussian histogram test (GHT) is asymptotically (in large samples) unbiased and efficient, that is the asymptotic expression for the variance of the estimator coincides with the *CRLB* in (4). The number of samples is related to the ADC resolution; each transition level should be excited several times. Therefore, for reasonable high resolution N will be high and the bias will be sufficiently small. However, these results presuppose Gaussian noise. In order to explore the effect of truncation a new expression for the *CRLB* has to be derived. Details are given in the Appendix, but the outlines are given below.

For an unbiased estimator of a scalar parameter T_k in a given model, the *CRLB* is obtained as [3]

$$CRLB(T_k) = \frac{1}{I(T_k)} \quad (5)$$

In (5), $I(T_k)$ is the Fisher information

$$I(T_k) = E\left[\left(\frac{\partial \ln p(x; T_k)}{\partial T_k}\right)^2\right] \quad (6)$$

where $p(x; T_k)$ is the probability mass function with respect to the binary observations $x[n]$, that is the stimuli signal $s[n]$ produces the observation $x[n]=0$ if $s[n] < T_k$ and $x[n]=1$ otherwise.

The stimuli signal for a fixed sample is $s[n] = T_k$, where T_k is Gaussian with variance σ^2 and mean value μ . A Bernoulli model can describe the estimation of a transition level T_k and $p(x; T_k)$ can be expressed as

$$p(x; T_k) = p_{1k}^x p_{0k}^{1-x}, \quad x = \begin{cases} 1 & s[n] \geq T_k \\ 0 & s[n] < T_k \end{cases} \quad (7)$$

The Fisher information is then (once again, see the Appendix for details)

$$I(T_k) = \frac{\frac{1}{\sigma^2 c^2} \phi\left(\frac{T_k - \mu}{\sigma}\right)^2}{F\left(\frac{T_k - \mu}{\sigma}\right) \left(1 - F\left(\frac{T_k - \mu}{\sigma}\right)\right)} \quad (8)$$

At this point in the derivation it is easy to see the relationship to an untruncated Gaussian noise. $F(\cdot)$ equals $\Phi(\cdot)$ for Gaussian noise and $c = 1$. Inserting that in the Fisher information above it is easy to see that the parallel to (4). It can also be mentioned that $c \rightarrow 1$ as σ decreases. In other words, if the variance reduces enough the *CRLB* for truncated noise goes to the *CRLB* for the untruncated noise. However, that signal will not be useful for this purpose, since the outermost values are not excited enough.

Finally, the *CRLB* is

$$CRLB(T_k) = \frac{\sigma^2 \left[c^2 - \operatorname{erf}\left(\frac{T_k - \mu}{\sqrt{2}\sigma}\right)^2 \right]}{4N \phi\left(\frac{T_k - \mu}{\sigma}\right)^2} \quad (9)$$

In reality, the generated noise signal $s[n]$ will be affected from external noise $w[n]$. It is well known that the sum of two independent and identically distributed Gaussian random variables has a Gaussian distribution with variance equal to the sum of the individual variances. Hence, measurement disturbance only induces a gain error, which will increase the *CRLB*. An explicit result for the *CRLB* subject to measurement noise is obtained by replacing σ^2 in (9) with $\bar{\sigma}^2$, where $\bar{\sigma}^2 = \sigma^2 + \sigma_{\text{Noise}}^2$ with σ_{Noise}^2 being the variance of the measurement noise.

3. Results

Figure 1 shows the *CRLB* as a function of the transition levels for different values of standard deviation ($\sigma = [0.5, \dots, 1]$). The number of samples must be sufficiently high in order to ensure excitation of all transition levels, which is a necessary condition for the existence of the *CRLB* [5]. In this analysis $N=1000$ samples is used. Full scale is chosen to unity. To illustrate the effect of truncation the *CRLB* corresponding to untruncated noise (4) is presented in the same diagram of Figure 1.

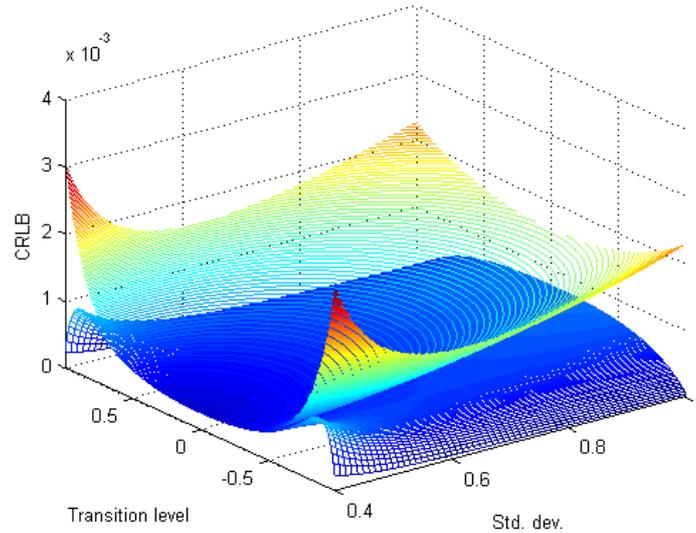


Figure 1: The *CRLB* as a function of transition level and standard deviation for Gaussian stimuli (upper surface) and truncated Gaussian (lower surface).

As can be seen from the figure the truncation results in a lower *CRLB*. This means that superior accuracy is expected for a given number of samples, or that fewer samples are needed to achieve a given accuracy, as long as N is large enough to excite all transition levels. The *CRLB* is also less depending on transition level for a truncated signal, which is the flatter *CRLB* surface in Figure 1, corresponds to the truncated signal. The variation of the *CRLB* over T_k is less for the truncated signal. Furthermore, one can find a variance that minimizes the *CRLB*. In Figure 2 the *CRLB* mean value of all transition levels for different standard deviations are presented. That is one possible parameter that can be used to decide a suitable standard deviation. Another approach is to look at the worst case; that is the maximum value of all transition levels.

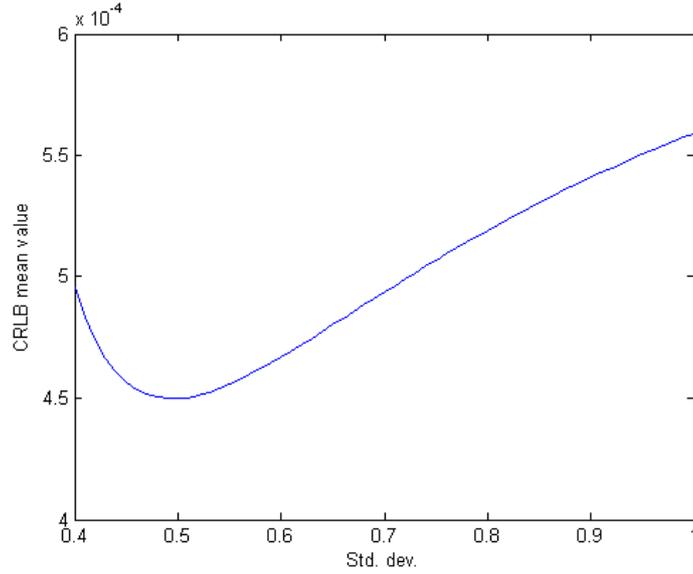


Figure 2: The average (with respect to T_k) *CRLB* as function of noise stimuli standard deviation σ .

4. Conclusions

When using Gaussian noise as stimulus signal for histogram tests, the signal has to be truncated to remove values outside the ADC full scale. The test sequence should be generated off-line; this not only minimizes the length of the sequence, but it will also increase the performance of the test. The Cramér-Rao Lower Bound (*CRLB*) is decreased, which gives better ratio between accuracy and sample length. The variation in *CRLB* for different transition levels decreases. It is shown that there exists a variance that minimizes the *CRLB*. This is the variance that should be used in test to improve accuracy or reduce the sample length. In the considered example with the transition levels in $[-1,1]$, the noise stimuli standard deviation corresponding to the minimum value in Figure 2 approximately is given by 0.5, that is roughly 4.5 % of the samples in a Gaussian sequence is removed in order to produce the truncated Gaussian sequence. If the standard deviation is chosen to minimize the maximum *CRLB* of all transition levels the sequence is reduced with approximately 9.5 %.

Appendix A

In this appendix the *CRLB* in (9) is derived. The derivations follow the derivations in [4] closely, where the Gaussian case was considered. Here, a truncated pdf is considered which will give a slightly different *CRLB*. For an unbiased estimator of a parameter T_k in a given model, the *CRLB* is obtained as [3]

$$CRLB(T_k) = \frac{1}{I(T_k)}$$

where $I(\theta)$ is the Fisher information

$$I(T_k) = E \left[\left(\frac{\partial \ln p(x, T_k)}{\partial T_k} \right)^2 \right]$$

The estimation of a transition level T_k can be described by a Bernoulli model, because the stimulus is a sequence of independent Gaussian random variables, and all the ADC input samples have the same probability p_k of not exceeding T_k gives by (3). As Bernoulli models are regular, the Fisher information associate to sample size of size N is N times the Fisher information of a unitary sized sample, that is

$$CRLB(T_k) = \frac{1}{NI(T_k)}$$

where $I(\theta)$ is the Fisher information of a unitary sized sample. For such a sample, only two events are possible, depending if the sample does or does not exceed the threshold, characterized respectively by the probabilities of occurrence p_{0k} and p_{1k} where

$$p_{0k} = 1 - F\left(\frac{T_k - \mu}{\sigma}\right)$$

$$p_{1k} = F\left(\frac{T_k - \mu}{\sigma}\right)$$

Consequently, $p(x; T_k)$ can be expressed as

$$p(x; T_k) = p_{1k}^x p_{0k}^{1-x}, \quad x = \begin{cases} 1 & s[n] \geq T_k \\ 0 & s[n] < T_k \end{cases}$$

The logarithm of $p(x; T_k)$ yields

$$\ln p(x; T_k) = (1-x) \ln p_{0k} + x \ln p_{1k}$$

Inserting the above expression into the expression for the Fisher information gives

$$I(T_k) = E \left[\left(\frac{1}{p_{0k}} \frac{\partial p_{0k}}{\partial T_k} (1-x) + \frac{1}{p_{1k}} \frac{\partial p_{1k}}{\partial T_k} x \right)^2 \right]$$

Further, we make use of the following result

$$\frac{\partial p_{1k}}{\partial T_k} = -\frac{\partial p_{0k}}{\partial T_k} = \frac{1}{c\sigma} \phi\left(\frac{T_k - \mu}{\sigma}\right)$$

The calculation of the Fisher information is then straightforward

$$I(\theta) = E \left[\frac{1}{\sigma^2 c^2} \phi\left(\frac{T_k - \mu}{\sigma}\right)^2 \left(\frac{x}{F\left(\frac{T_k - \mu}{\sigma}\right)} - \frac{1-x}{1-F\left(\frac{T_k - \mu}{\sigma}\right)} \right)^2 \right]$$

$$= \frac{1}{\sigma^2 c^2} \phi\left(\frac{T_k - \mu}{\sigma}\right)^2 E \left[\left(\frac{x - F\left(\frac{T_k - \mu}{\sigma}\right)}{F\left(\frac{T_k - \mu}{\sigma}\right) \left(1 - F\left(\frac{T_k - \mu}{\sigma}\right)\right)} \right)^2 \right]$$

$$= \frac{1}{\sigma^2 c^2} \phi\left(\frac{T_k - \mu}{\sigma}\right)^2 \frac{1}{F\left(\frac{T_k - \mu}{\sigma}\right) \left(1 - F\left(\frac{T_k - \mu}{\sigma}\right)\right)}$$

In the last equality, variance of the Bernoulli model is used. Finally, $F(T_k)$ in (3) is inserted into the formula above. The result reads

$$I(T_k) = \frac{1}{\sigma^2 c^2} \phi\left(\frac{T_k - \mu}{\sigma}\right)^2 \left(\frac{1}{\left[\frac{1}{2} - \frac{1}{2c} \operatorname{erf}\left(\frac{T_k - \mu}{\sqrt{2}\sigma}\right) \right] \left[\frac{1}{2} + \frac{1}{2c} \operatorname{erf}\left(\frac{T_k - \mu}{\sqrt{2}\sigma}\right) \right]} \right)$$

$$= \frac{1}{\sigma^2 c^2} \phi\left(\frac{T_k - \mu}{\sigma}\right)^2 \left(\frac{1}{\left(\frac{1}{2}\right)^2 - \left[\frac{1}{2c} \operatorname{erf}\left(\frac{T_k - \mu}{\sqrt{2}\sigma}\right) \right]^2} \right) = \frac{\frac{4}{\sigma^2} \phi\left(\frac{T_k - \mu}{\sigma}\right)^2}{c^2 - \left[\operatorname{erf}\left(\frac{T_k - \mu}{\sqrt{2}\sigma}\right) \right]^2}$$

The *CRLB* is then

$$CRLB = \frac{1}{NI(\theta)} = \frac{\sigma^2 \left[c^2 - \operatorname{erf}^2\left(\frac{T_k}{\sqrt{2}\sigma}\right) \right]}{4N \phi\left(\frac{T_k - \mu}{\sigma}\right)^2}$$

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